Varying the Number of Signals in Matching Markets[⋆]

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Abstract. In large matching markets between job candidates and organizations, organizations may be unable to effectively identify interested candidates due to a large volume of applications. The resulting congestion makes it unlikely for candidates to receive offers from their most preferred organizations, leading to significant mismatch. We study how signaling mechanisms can be used as a market design tool to reduce the congestion in such markets. Specifically, we look at how the number of signals available to market participants affects welfare and the number of matches using a large market model. We show that for sufficiently many signals, candidate welfare and the number of matches decrease as a function of the number of signals, while the behavior of organization welfare depends on the extent to which organizations value top candidates. Furthermore, we describe a class of firm utility functions for which these limiting effects start to hold at realistic numbers of signals S.

Keywords: Signaling · Matching · Large Markets

1 Introduction

In most matching markets, candidates apply to organizations and organizations choose a subset of applicants to accept. Since the quantity of offers is typically limited, organizations must not only determine candidates' qualities, but also discern whether candidates are realistically attainable. However, it is often easy for candidates to express interest in many organizations by sending out a large number of applications. This behavior may lead to market congestion, in which organizations become overwhelmed by the volume of applications and are unable to select for interested applicants (Coles et al., 2013; He and Magnac, 2017; Arnosti et al., 2014). Candidates are also unlikely to receive offers from their most preferred firms under such circumstances. As a result, there can be significant mismatch between candidates and organizations, as well as suboptimal welfare: most candidates fail to receive offers from their top choices, and organizations waste effort on recruiting uninterested candidates.

The introduction of a signaling mechanism is one strategy to reduce the amount of congestion and mismatch in such markets. A typical signaling mechanism in this context allows candidates to signal interest to organizations, but limits the number of signals that any candidate may send. Because the signals of this mechanism are scarce, sending a signal has a tangible opportunity cost. As a result, signaling becomes more than just "cheap talk" and can serve as a means for candidates to credibly convey interest towards organizations.

An important design consideration when implementing a signaling mechanism is selecting the number of signals that each candidate may send. If candidates can signal to all or a significant fraction of the firms, then signals lose their scarcity and may devolve into cheap talk. At the other extreme, if candidates do not have enough signals, then they may not be able to communicate much information about their preferences, and the signaling mechanism may not be as effective as possible. These opposing effects show that it is not obvious how to optimally set the number of signals, e.g., so that welfare metrics such as aggregate welfare or the number of matches are maximized.

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In this paper, we analyze how varying the number of signals available to candidates affects welfare in signaling markets. We consider a large market model in which a continuum of candidates is matched to a finite, discrete set of organizations. We then introduce a one-sided signaling mechanism that allows each candidate to signal to one or several organizations and examine how this mechanism affects market dynamics and statistics such as welfare and the number of matches. Our main result is a characterization of how the number of signals affects welfare metrics in the limiting case of many signals. We show that for sufficiently many signals, candidate welfare and the number of matches decrease as a function of the number of signals, while the behavior of organization welfare depends on the extent to which organizations value top candidates.

Signaling as a market design technique has been studied empirically in both the economics job market and in online matchmaking. In the economics job market, the AEA implements a mechanism through which job seekers are permitted to "signal" to up to two prospective employers. Signaling has been observed to increase the probability that a candidate lands an interview for one of their highly ranked positions and to allow for better expression of idiosyncratic preferences, e.g., over school type and over location (Coles et al., 2010). Lee and Niederle (2015) make similar empirical observations about an online matchmaking market where candidates were randomly endowed with either two or eight virtual roses. The introduction of the signal was observed to increase the total number of matches as well as increase the probability a proposal was accepted. Although these empirical studies demonstrate the benefits of a market with signals over a market without signals, there has not yet been a systematic analysis of how varying the number of available signals between different nonzero values affects market welfare.

We fill in this gap from a theoretical perspective by analyzing a large market model of signaling. In our model, a continuum of candidates gets matched to a finite, discrete set of firms, and candidates may express interest towards firms through a signaling mechanism with scarce signals. We show that a non-babbling equilibrium, i.e., an equilibrium in which firms respond to signals, always exists in this game. Furthermore, if we restrict our attention to symmetric, anonymous equilibria, a non-babbling equilibrium occurs if and only if each candidate signals to exactly their top choice firms.

Our main result is a characterization of how the number of signals affects welfare metrics once there are sufficiently many signals. In this regime, we show that a unique symmetric, anonymous, non-babbling equilibrium exists. This equilibrium is such that candidates use all of their signals and organizations only consider candidates who have signaled. For these equilibria, we show that increasing the number of signals decreases worker welfare and the number of matches, since it becomes increasingly difficult to express preferences. On the other hand, organizations may or may not prefer more signals. If organizations highly value top candidates, then they would prefer weaker signaling so that they can pursue their most preferred candidates, whereas if organizations are indifferent between strong candidates and very strong candidates, then they would prefer stronger signaling so they can pursue the candidates most interested in them.

Finally, we consider our results in the context of more explicit parameter settings. We characterize a class of firm utilities for which these limiting results start to hold when the number of signals S is at least 4. These utility functions show that although our results are for "sufficiently many" signals, they do hold for very realistic values of S. In the other direction, we describe a market with a relatively simple firm utility function that demonstrates the indeterminacies and multiple equilibria inherent for few signals. This example also concretely illustrates the potential of using many signals to achieve welfare improvements.

Past theoretical work on matching markets with a centralized signaling mechanism includes Coles et al. (2013) and Kushnir (2013). Coles et al. (2013) study signaling in the context of small, symmetric markets. They model signaling as a Bayesian game and characterize the set of equilibrium strategies. Furthermore, they show that under the assumption of uniform preferences, the introduction of a signal increases the utility of applicants and the expected number of matches, but has an indeterminate effect on the utility of organizations. While the uniform preferences assumption may not accurately model all real-life settings, it is crucial in making the analysis tractable: Coles et al. (2013) remark that even when their model is extended to multiple blocks of firms with block-uniform preferences, the welfare comparisons across different equilibria become indeterminate. In another direction, Kushnir (2013) observes that when there is heterogeneity among applicants and preferences are almost completely specified, the introduction of a signaling mechanism with one signal may improve the expression of idiosyncratic preferences, but may also hurt more "mainstream" applicants on average. These previous works show that for several market settings, signaling can allow for the better expression of preferences for various segments of the market.

We now turn to the question of how to maximize the usefulness of signals by varying their number. We borrow the game structure and uniform preferences assumptions of Coles et al. (2013) in our model. However, Coles et al. (2013) do not have the necessary architecture in their model to study the impact of changing the number of signals to different nonzero values, since they focus on comparing the game with signals to the game with no signals. Nonetheless, we give a generalization of their model to any nonzero number of signals, with the modification of shifting to a large market setting in order to make studying the effects of varying the number of signals tractable. Large market models have been previously considered by Azevedo and Leshno (2016) in the contexts of stable matchings. In their work, the continuum matching economy serves as an analytic tool to enable comparative statistics to be taken. We similarly use the continuum assumption to allow us to analyze the impact of varying the number of signals.

The remainder of this paper is structured as follows: In §2, we present and formalize our large market model of a one-sided signaling mechanism as a Bayesian game. In §3, we state our solution concept and characterize the equilibrium strategies in our game. In §4, we investigate the signaling dynamics of our game with signals and characterize how varying the number of signals affects welfare metrics (number of matches, worker welfare, and firm welfare) in the limiting case. Proofs and useful examples are included in the appendix.

2 Model

Our model captures the behavior of firms and workers in a large matching market with a signaling mechanism. We extend the approach of Coles et al. (2013) to allow for more than one signal, making similar symmetry and uniformity assumptions on the market participants. We also shift the model to a large market setting to make our analysis of varying the number of signals tractable.

We assume the market consists of a finite number of firms and a continuum of workers. In this market, we make the standard Bayesian game assumption that the *distribution* of firm preferences over workers and worker preferences over firms is common knowledge. Furthermore, each worker knows their own preference ordering over the firms, and each firm knows the utility that it will derive from each worker, but each worker has no knowledge of how firms evaluate them relative to other workers, and each firm has no knowledge of how any given worker ranks it relative to other firms. To make our analysis tractable, we assume firms independently assign scores to workers, so that firm preferences over workers are uncorrelated, and we assume workers' preference orderings over firms are drawn uniformly at random, so that there is no common ranking of the firms.[1]

[1] While Coles et al. (2013) also study the case of multiple blocks of firms, some of their welfare trends become indeterminate for more than one block of firms in their model. For this reason, we focus on the case of one block in our paper.

Finally, we assume that firms' preference orderings over workers and workers' preference orderings over firms are independent of each other.

We model the matching market in three stages. We start with a signaling stage, during which each worker *signals* to up to a fixed number of firms through the signaling mechanism. Each signal is binary and does not transmit any further information. Then, each firm, having received the set of workers who have signaled to it, sends offers to a fixed quantity of workers. Finally, each worker accepts their most preferred offer (if they have any offers). We assume that all workers share a common utility function and that all firms share a common utility function. Rejected offers yield 0 utility, firm utility from an accepted offer is purely determined by its score on the worker, and worker utility from an accepted offer is purely determined by their ranking of the firm.

Modeling the workers as a continuum greatly simplifies the space of possible equilibrium firm strategies. This assumption has a convexifying effect on the game (Aumann, 1964), since no individual worker is able to influence the measure of workers who signal to a given firm. Furthermore, by a line of work on exact laws of large numbers for atomless measure spaces (Sun and Zhang, 2009), we can assume the workers belong to an atomless measure space where such a law of large numbers holds. Stripping atomicity from the workers and assuming exact convergence to the distribution over preferences resolves complications imposed by the model of Coles et al. (2013), in which the firm strategies took into account the number of signals received by the firm. The continuity of the worker types thus simplifies the equilibrium structure, allowing us to apply analytic tools and take comparative statics that would have otherwise been intractable in the discrete setting.

We describe the distribution of preferences in §2.1 and the stages of the game in §2.2.

2.1 Distribution of Preferences and Utilities

Let $\mathcal{F}$ denote the set of firms, and let $F := |\mathcal{F}|$ be the number of firms. We use $\mathcal{S}_{\mathcal{F}}$ to denote the set of permutations of $\mathcal{F}$. Let $\mathcal{W} := \mathcal{S}_{\mathcal{F}} \times [0,1]^F$ be the continuum of worker types. That is, worker rankings of firms are given by permutations drawn from $\mathcal{S}_{\mathcal{F}}$, which correspond to a total ordering of firms from highest to lowest ranked. Firm rankings of workers are given by scores in $[0,1]$, where e_f^w denotes the score of worker w given by firm f. The set of all firm scores for a given worker can thus be represented as an element of $[0,1]^F$. Finally, let $S \geq 0$ be the maximum number of signals that each worker is allowed to send.

We assume that the distribution of the worker types over $\mathcal{W}$ is given by the product of a distribution over $\mathcal{S}_{\mathcal{F}}$ of worker preferences over firms and a distribution over $[0,1]^F$ of firm preferences over workers. This product construction yields the key property that the workers' preferences over firms and the firms' preferences over workers are independent of each other. The workers themselves (and the corresponding firm preferences over workers) will then compose an atomless measure space with types drawn uniformly from $\mathcal{W}$, such that the distribution of worker types is exactly the uniform distribution over $\mathcal{W}$ (Sun and Zhang, 2009).

By assuming that the distribution of firm preferences over workers is the uniform distribution over $[0,1]^F$, we make firm preferences over any given worker independent. Nonetheless, observe that there are some workers who have high scores from all the firms (universally "high-performing" workers), some workers who have high scores from some firms but not other firms (workers endowed with some skills but lacking other skills), and some workers who have low scores from all the firms (universally "low-performing" workers).

The uniform preferences assumption has been used throughout the signaling literature, for example in the one-signal model of Coles et al. (2013) (which our model extends) to make their welfare analyses determinate. In fact, Coles et al. remark that even when their model for one signal is extended to multiple blocks of firms with block-uniform preferences, the welfare comparisons across different equilibria become indeterminate. When a single firm responds to more signals, firms in lower-ranked blocks may benefit so that there is no longer a purely negative spillover on other firms. We make the same uniform preferences assumption to make possible our analogous results on the dynamics of the signaling game and our analysis of welfare metrics in the limit.

Our welfare comparison theorems rely on the property that there is a unique equilibrium in the limit. The uniqueness comes from the fact that not receiving a signal from a worker indicates that the firm is not high on the worker's preference list. This fact, however, can fail if workers' preferences are correlated, e.g., workers may prefer to signal to lower tier firms so that their signal will carry more weight. While such strategies may be of interest in practice, it nonetheless makes the space of equilibria, which may now depend heavily on the number of signals, intractable to analyze across different numbers of signals.

We also assume that firms share a common utility function $u\colon [0,1] \to \mathbb{R}_{\geq 0}$ mapping worker score to utility. If a firm f's offer is rejected by a worker, then the firm receives 0 utility. Otherwise, if worker w accepts firm f's offer, the firm receives utility equal to $u(e_f^w)$. Naturally, the utility function u should be increasing, i.e., workers with higher score yield higher utility. For technical reasons, we also assume that u is continuously differentiable and strictly increasing. Observe that by setting u to be a quantile function, we may have $u([0,1])$ be distributed as any bounded distribution $\mathcal{D}$ on $\mathbb{R}$ whose quantile function is continuously differentiable and increasing. Using this construction, we can convert the uniform distribution over $[0,1]^F$ of worker scores to any such product distribution $\mathcal{D}^F$ over $\mathbb{R}_{\geq 0}^F$ of utilities derived from workers.

We assume that the distribution of worker preferences over firms is the uniform distribution over S_F, so that firms are equally ranked on average. Here, workers share a common utility function $v\colon \{1,\ldots,F\} \to \mathbb{R}_{\geq 0}$ mapping firm rank to utility that is a decreasing function. If a worker does not receive offers from any firm, they derive 0 utility from the game. Otherwise, a worker derives $v(i)$ utility if they accept the offer of their i-th highest ranked firm.

2.2 Stages of the Game

We model the signaling market as a game with three stages. We assume that the distribution over worker types is common knowledge, but that at the beginning of the game, firms do not have any knowledge of any given worker's preferences, and the workers do not have any knowledge of any specific firm's preferences. In other words, each firm and each worker knows only their own preferences. This assumption ensures that the only information communicated between the firms and the workers is through the presence (or lack) of the binary signal.

To define the game, we introduce a new constant $0 \leq \gamma < 1/F$, that roughly quantifies the competitiveness of the market by serving as an upper bound on the number of offers each firm is allowed to send. The constraints $0 \leq \gamma < 1/F$ ensure there are fewer positions available than workers. We require competition since when $\gamma \geq 1/F$, each worker is able to be matched to their preferred firm, leading to the existence of non-competitive equilibria in which workers only signal to their most preferred firm.

The game proceeds in the following three stages:

- During the first stage, each worker, knowing only their own preferences over firms, sends signals to up to S different firms for some fixed parameter S.
- At the start of the second stage, each firm is notified of the set of workers that signaled to it. Given the signals that it receives as well as its ranking of the workers by score, each firm then sends offers to a measurable subset of the workers of measure at most γ.
- In the third and final stage, each worker accepts the offer from the highest ranked firm from whom they receive an offer, if they receive any offers at all.

The offer structure of our game is based on the one-block, one-signal game of Coles et al. (2013).[2] We allow workers to send fewer than S signals, so that workers do not have to signal if signaling lowers their probability of acceptance, thus preventing signals from becoming a negative commodity for the workers.

3 Equilibrium Strategies

Our solution concept is perfect Bayesian equilibrium restricted to strategies that are symmetric and anonymous with respect to each side of the market. The symmetry assumption comes from the fact that types on each side are drawn from the same distribution. Anonymity is necessary to rule out strategies in which there is unrealistic coordination outside of the signaling mechanism. These assumptions on strategy profiles are similar to those of related works (Coles and Shorrer, 2014; Coles et al., 2013).

We say that a worker strategy is *anonymous* if the strategy only depends on the preference profile of the worker, i.e., how a firm is treated depends only on its rank. This corresponds to the traditional definition of an anonymous strategy: a worker strategy σ_w is anonymous if for any permutation π of the firms and any preference profile θ_w, we have that $\sigma_w(\pi(\theta_w)) = \pi(\sigma_w(\theta_w))$. We say that a worker strategy profile is *symmetric* if each worker has the same response to a given preference profile over the firms. Next, given that a firm is facing a symmetric strategy profile from the workers, which in particular gives us a well-defined measure for the set of workers who signal to any given firm, we say that a firm strategy is an *anonymous response* if the strategy depends only on whether or not the worker signaled and the firm's own score on the worker. Finally, we say that a firm strategy profile is *symmetric* if each firm has the same response to a given preference profile and a given set of workers who signaled.

In order to determine the set of symmetric, anonymous equilibria of the signaling game, we first consider firm strategies in equilibrium. Recall that worker preferences over firms are independent from firm preferences over workers. Intuitively, given this independence assumption, the optimal firm strategy should be specified by *cutoffs*, in which the firm sends offers to all signaling workers with scores above a certain cutoff and fills the remainder of its quota with non-signaling workers with scores above another cutoff; this motivates the following definition of *cutoff strategy*:

Definition 1. *A* cutoff strategy *is a firm strategy given by cutoffs c_S and c_N such that the firm sends offers to all signaling workers whose scores is at least c_S and all non-signaling workers whose score is at least c_N. If the set of signaling workers is empty, then we take c_S to be 0; likewise, if the set of non-signaling workers is empty, then we take c_N to be 0.*

Cutoff strategies are clearly anonymous. We show that for any nonzero number of signals, firms play cutoff strategies in equilibrium. For this reason, we will only consider cutoff strategies for the remainder of our discussion.

Lemma 1. *In any symmetric, anonymous equilibrium of the game with a nonzero number of signals, firms play cutoff strategies (or measure 0 deviations from cutoff strategies) and all firms have the same cutoff c_S.*

[2] In Coles et al. (2013), each firm is only allowed to make an offer to at most one worker, and each worker is allowed to accept at most one offer from at most one firm. In our model, we maintain that each worker can accept at most one offer. However, we instead set the maximum measure of offers that a firm is permitted to send to γ. While this offer structure does not precisely model the dating market (where candidates can "accept" multiple dates) or college admissions (where colleges can select the number of students to accept based on yield rates), the simplicity of the game, in both the setting of Coles et al. (2013) and in our setting, enables the analysis to be tractable.

Lemma 1 relies on the independence of worker rankings of firms from firm rankings of workers, which guarantees that the probability a worker accepts an offer is independent of the worker's score. Without this assumption, each firm may be able to strategize based on the workers' ranking of it. For example, if one firm were universally much lower ranked than the other firms, it may choose not to waste offer slots on very talented workers. As a result, our model applies to a segment of the job market in which firms are of similar caliber and workers are of similar caliber, where our independence assumptions are reasonable.

The equilibrium worker strategy depends on how c_S compares to c_N. If $c_S = c_N$, then firms effectively ignore signals, giving rise to a babbling equilibrium. Otherwise, if $c_S < c_N$, then due to the symmetry between the firms, we show that it is optimal for each worker to truthfully signal to their top S choices.

Lemma 2. *The following are the only two possibilities for symmetric, anonymous worker strategy profiles in equilibrium:*

1. *If signaling increases the probability of receiving an offer (i.e., $c_N > c_S > 0$), then workers always signal to their top S firms.*
2. *Otherwise, if firms ignore signals (i.e., $c_S = c_N$), then any symmetric, anonymous worker strategy profile leads to a babbling equilibrium.*

In particular, $c_S > c_N$ is not possible in equilibrium, since workers will never signal to a firm whose cutoff for non-signaling workers is higher than its cutoff signaling workers.

For the remainder of the paper, we focus on the equilibria where the welfare metrics change as a function of the number of signals. In the first scenario, where $c_S = c_N$, the game is in a babbling equilibrium and the welfare metrics are equivalent to the equilibrium in the game without signals. For this reason, we focus on the non-babbling equilibria.

The simplicity of worker strategies depends on the fact that the distribution of firm preferences is uniform. Without this assumption, optimal worker strategies are much less clear: the symmetry assumptions on the equilibria are no longer reasonable, and optimal strategies can no longer be parameterized by a single value. While the strategy space in our model does not capture the full range of behavior that could occur in practice, its simplicity makes our analysis of welfare metrics with varied numbers of signals tractable.

Given that the distribution of worker preferences is uniform over $\mathcal{S}_{\mathcal{F}}$, we show that there always exists at least one non-babbling equilibrium in symmetric and anonymous strategies for the signaling game.

Theorem 1. *For each firm utility function u and for any nonzero number of signals, there exists a non-babbling equilibrium in symmetric and anonymous strategies for the signaling game where workers signal to their top S firms.*

We show in Appendix A that there exist firm utility functions u that lead to multiple non-babbling equilibria where workers signal to their top S firms. In §4, we will analyze the welfare metrics (worker welfare, firm welfare, and number of matches) at these multiple equilibria.

In non-babbling equilibria in symmetric and anonymous strategies, we know by Lemma 2 that each worker signals to their top S firms. In Appendix B, we show the reverse implication: for any symmetric, anonymous equilibrium, if workers signal to their top S firms, then $c_S = c_N$, i.e., where signaling does not change the probability of receiving an offer, is never an equilibrium. For this reason, we focus for the remainder of the paper on the symmetric, anonymous equilibria that arise when each worker signals to their top S firms.

4 Impact of Signals

Having characterized the non-babbling equilibria in symmetric and anonymous strategies, we now study the welfare metrics associated to the signaling game. In §4.1, we discuss the signaling dynamics of the game with signals. In §4.2, we discuss how varying the number of signals affects the welfare metrics.

4.1 Signaling Dynamics

We now consider the dynamics of the game with signals. As we showed in Theorem 1, there is always at least one non-babbling equilibrium in the game with signals, and as we show in Appendix A, there exist firm utility functions u that lead to multiple symmetric, anonymous, non-babbling equilibria in the signaling game. Thus, it is of interest to consider how the welfare metrics compare between the multiple equilibria that arise for a fixed number of signals.

We show that the welfare metrics of firm welfare, worker welfare, and number of matches are all monotonic in the cutoff value c_S parameterizing the equilibrium. Namely, there is an opposition of interests between firms and the workers/number of matches: at equilibria corresponding to lower cutoffs, firm welfare is lower, worker welfare is higher, and the number of matches is higher.

Theorem 2. *For any nonzero number of signals, suppose there exist symmetric, anonymous, non-babbling equilibria with cutoffs at $c_S = c_1$ and $c_S = c_2$, respectively, such that $c_1 < c_2$. Then at the equilibrium with cutoff $c_S = c_1$, firm welfare is lower, worker welfare is higher, and the number of matches is higher.*

The cutoff parameter c_S can be thought of as the extent to which firms respond to worker signals. This observation provides an explanation of the monotonicity results for firm and worker welfare: at lower cutoffs, firms consider worker preferences more, which is beneficial to workers and harmful to firms. Finally, the number of matches is intuitively decreasing in the cutoff since signaling reduces the amount of congestion in the market.

We also investigate how welfare metrics compare between equilibria in the game with signals and in the game without signals. We specifically show that the game with signals is always preferable to the game without signals with respect to worker welfare and the number of matches. The change in firm welfare is indeterminate.

Theorem 3. *For any nonzero number of signals, both worker welfare and the number of matches are greater at any equilibrium in the game with signals than in the game without signals.*

Since signaling forces firms to consider worker preferences, any amount of signal intuitively should improve worker welfare, which is exactly what occurs in this model. For a similar reason, the number of matches should intuitively increase with signaling. The change in firm welfare is indeterminate because the introduction of a signaling mechanism has two opposing effects. The first effect is a reduction in the market power of firms from taking into account worker preferences, which decreases firm welfare; the second is a partial resolution of the coordination problem which reduces the waste of multiple firms sending offers to the same worker.

These properties of our model for signaling markets, which hold for any nonzero number of signals, capture and reinforce many of the same structural properties as the one-block model with one signal studied by Coles et al. (2013).[3] However, unlike in their model, our firm strategies are specified by one cutoff as a result of having a continuum of workers, so we can analyze the effect of varying the number of signals on these welfare metrics by taking comparative statics.

[3] One difference between the dynamics of our model and the dynamics of the model of Coles et al. (2013) is whether $c = 1 - \gamma$ is an equilibrium if worker signal truthfully, while in our model, for any $S < F$, the cutoff $c = 1 - \gamma$ is

4.2 Welfare Metrics in the Limit

Our main goal is to investigate how firm welfare, worker welfare, and the number of matches change as a function of the number of signals S available to each worker. One complication that arises is the possibility of multiple equilibria in this game. In Appendix A, we present a firm utility function for which the number of equilibria and the cutoff values at the equilibria change as the number of signals varies. Furthermore, although we have shown in the previous section that welfare metrics are monotonic in the cutoff for a fixed number of signals, it is indeterminate how these values compare across different numbers of signals, as we also demonstrate through our example in Appendix A.

We resolve these complications by focusing on the relevant welfare metrics in the case where there are many signals. The regime of interest is where the signals transmit enough information to have a significant impact on firm strategy, but not enough so that signals devolve into cheap talk. Our next theorem shows that given sufficiently many signals, there exists only one symmetric, anonymous, non-babbling equilibrium, in which workers signal to their top S firms and firms only send offers to signaling workers. This phenomenon occurs when there are enough signals that *not* sending a signal becomes a strong negative indication of interest. The uniqueness of this equilibrium will allow us to take comparative statics to analyze the effect of increasing the number of signals on welfare metrics.

Theorem 4. *Suppose that $\gamma = \frac{\alpha}{F}$ for some constant $\alpha < 1$. For each utility function u, there exists a threshold value $C < \infty$ such that for any $S \geq C$ and $F > S$, there exists only one symmetric, anonymous, non-babbling equilibrium in the signaling game with F firms and S signals. This equilibrium occurs when firms only make offers to signaling workers, i.e., when firm strategies have cutoffs $c_S = 1 - \frac{\alpha}{S}$ and $c_N = 1$.*

When a large number of signals are available to workers, workers are able to signal to enough firms that not receiving a signal from a worker strongly indicates to a firm that it is low in the worker's preference ranking. In particular, the probability that a worker is rejected by all of its signaled-to firms and accepts the offer of a non-signaled-to firm becomes low. In contrast, in the case where there are very few signals, workers only signal to their top few firms, so it may still be worthwhile for firms to extend offers to highly valued workers who have not signaled. In this case, it is indeterminate whether $c_S = 1 - \frac{\alpha}{S}$ is an equilibrium, and there may also be other equilibria, as we show in Appendix A.

Now that we have shown that, given sufficiently many signals, the unique equilibrium is at $c_S = 1 - \frac{\gamma F}{S}$ and $c_N = 1$, our task boils down to studying how the welfare metrics change as the number of signals increases. We show that worker welfare and the number of matches decrease as the number of signals increases at this equilibrium.

Theorem 5. *Suppose that $\gamma = \frac{\alpha}{F}$ for some constant $\alpha < 1$. Let u be a utility function, and let C be a threshold value such for all $S \geq C$, the signaling game has only one symmetric, anonymous, non-babbling equilibrium at $c_S = 1 - \frac{\alpha}{S}$. (This threshold C is guaranteed to exist by Theorem 4.) Then, for any $S \geq C$ and $F > S$, worker welfare and the number of matches decrease as the number of the signals increases.*

The decrease in worker welfare and the number of matches occurs because firms already respond maximally to signals, and increasing the number of signals available to each worker without

never an equilibrium. On the other hand, in Coles et al. (2013), the cutoff $c = 1 - \gamma$ is always an equilibrium in the game with signals. The difference arises because in our model, firms send offers to more than one candidate, and the margin now matters. One consequence is that in our model, shifts to strategies based on signaling are more likely to endogenously occur given the existence of a signaling mechanism.

increasing the extent to which firms respond to signals dilutes the strength of the signal. Now, workers very interested in a firm may get their spots taken away by less interested workers. At the extreme, signaling devolves into cheap talk, and the market where the number of signals equals the number of firms is indistinguishable from the market without signals.

Now, we consider firm welfare. We show that, in the regime of many signals, the change in firm welfare as the number of signals increases is a local property of the utility function: namely, it depends solely on the rate of increase of u at the maximum worker score. We show that if this derivative is small, then firm welfare is decreasing in the number of signals in the limit, and if the derivative is large, then firm welfare is increasing in the number of signals in the limit.

Theorem 6. *Suppose that $\gamma = \frac{\alpha}{F}$ for some constant $\alpha < 1$. Let u be a utility function, and let C be such that for all $S \geq C$, the game with signals has only one symmetric, anonymous non-babbling equilibrium at $c_S = 1 - \frac{\alpha}{S}$. (This threshold value is guaranteed to exist by Theorem 4.) There exists a threshold value $C' \geq C$ that satisfies the following property:*

- *If $u'(1) < \frac{\alpha u(1)}{e^{\alpha}-1}$, then for any $S \geq C'$ and $F > S$, firm welfare in the signaling game with utility function u, S signals, and F firms is a decreasing function of S. Furthermore, firms prefer any number of signals $S \geq C'$ to no signals.*
- *If $u'(1) > \frac{\alpha u(1)}{e^{\alpha}-1}$, then for any $S \geq C'$ and $F > S$, firm welfare in the signaling game with utility function u, S signals, and F firms is an increasing function of S. Furthermore, firms prefer no signals to any number of signals $S \geq C'$.*

In an equilibrium where firms only send offers to signaling workers, firms face a tradeoff as the number of signals increases between being able to send offers to a stronger set of workers and getting offers rejected by very strong workers who receive multiple offers. The sign of this comparative static therefore hinges on the extent to which firms value stronger workers. If firms significantly value stronger workers, then firms would prefer more signals, i.e., to be closer to the case where signals are cheap talk. Otherwise, firms would prefer fewer signals in order to reduce the number of offers received by workers that are highly rated by many firms. In the limit, how much firms value stronger workers is captured by the derivative $u'(1)$.

When the number of signals is small, however, the effect of increasing the number of signals available has an indeterminate effect on firm utility, even if we assume that the game is in the equilibrium where firms only make offers to signaling workers. The direction of change in firm utility depends on the marginal gain of accepting a worker at the cutoff for signaling workers. If this marginal gain at the cutoff is very different from the marginal gain of accepting a top worker, then the direction of change may be indeterminate. In Appendix A, we construct an example in which the direction of change is indeterminate when S lies between the bounds C and C'. (That is, it is necessary to allow threshold C' to be greater than threshold C.)

Combining the previous two theorems yields the following result about total firm and worker welfare. If $u'(1)$ is small, then total welfare is a decreasing function of S for sufficiently large S, and if $u'(1)$ is large, then there is an opposition of interests between firms and workers. In the former case, the optimal number of signals should therefore not be too high, since the limiting behavior has negative consequences for both firms and workers. In the latter case, there exists an opposition of interests between firms and workers in the regime of many signals, and the optimal amount of signaling in this case becomes dependent on the method of welfare aggregation.

Corollary 1. *Suppose that $\gamma = \frac{\alpha}{F}$ for some constant $\alpha < 1$. Let u be a firm utility function, and assume that $u'(1) \neq \frac{\alpha u(1)}{e^{\alpha}-1}$. Let C' be the threshold value such that there is a unique symmetric, anonymous, non-babbling equilibrium at $c_S = 1 - \frac{\alpha}{S}$ and such that the change in firm welfare as the* *number of signals increases is monotonic. (This threshold is guaranteed to exist by Theorem 4 and Theorem 6.) Suppose that $S \geq C'$ and $F > S$. If $u'(1) < \frac{\alpha u(1)}{e^{\alpha}-1}$, then total welfare is a decreasing function of S. And if $u'(1) > \frac{\alpha u(1)}{e^{\alpha}-1}$, then there is an opposition of interests between firms and workers.*

Since these results are limiting results, this raises the question of how large of the thresholds C and C' needs to be for well-behaved firm utility functions. We show that for utility functions that grow sufficiently slowly between 0.75 and 1, firm welfare and total welfare are decreasing functions of S when $S \geq 4$.

Lemma 3. *Suppose that $\gamma = \frac{\alpha}{F}$ for some constant $0.8 \leq \alpha < 1$. Suppose that u satisfies $u(y) \geq u(1)(0.76 + 0.24y)$ for $1 \geq y \geq 0.75$. Suppose that $F > S \geq 4$. Then, there is only one symmetric, anonymous, non-babbling equilibrium in the signaling game with S signals and firm utility function u. Furthermore, firm and worker welfare are both decreasing functions of S for $S \geq 4$.*

This shows for certain utility functions, the optimal number of signals is between 0 and 4, demonstrating that limiting results can take effect for reasonable S.

5 Conclusion

Scarce signaling mechanisms are a useful market design tool to reduce congestion and mismatch in markets where credible communication of preferences is difficult. Such mechanisms have been implemented in practice in the economics job market and in the context of matchmaking in online dating. However, the number of signals available to each candidate varies between mechanisms. Understanding how to best set the number of signals available is therefore a key design question for the implementation of such signaling mechanisms.

In this paper, we developed a large market model for signaling markets and studied how varying the number of signals affects various welfare metrics. Assuming uniform preferences, we showed that in a symmetric, anonymous, non-babbling equilibrium, firms play cutoff strategies and workers signal to their top S firms. Our main result is that in the limit, there exists a unique equilibrium in which firms respond maximally to signals. At this equilibrium, increasing the number of signals decreases worker welfare and the number of matches, while the effect on firm welfare depends on the extent to which firms value top workers.

Future lines of research include relaxing the uniform preferences assumption and characterizing welfare behavior in the case of very small numbers of signals.

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Appendix

A Indeterminacy for Small Numbers of Signals

We show through a simple example that when the number of signals is small, the impact of varying the number of signals on welfare metrics is indeterminate. This indeterminacy illustrates that assuming a large number of signals is in fact necessary for our preceding results about the monotonic change of welfare metrics.

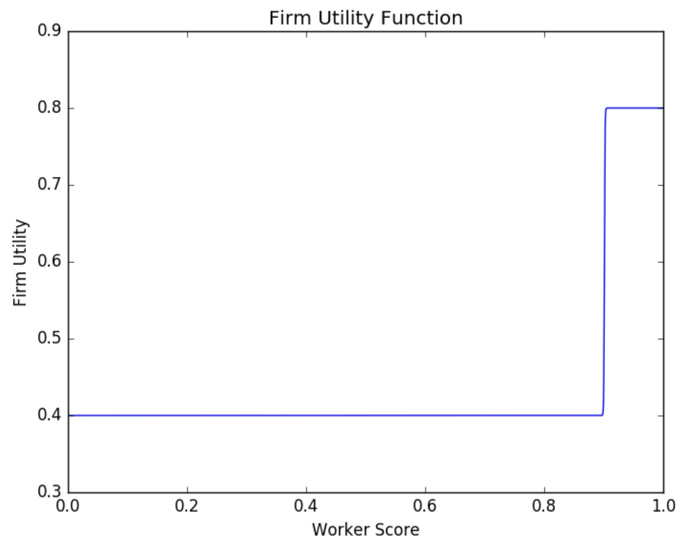


Fig. 1: Firm Utility Function u

We consider a signaling market of $F = 50$ firms and where $\alpha = 0.99999999$. In this market, firms view workers as belonging to either a top tier or bottom tier, with 10 percent of workers in the top tier and the remaining 90 percent in the bottom tier. That is, we define firm utility to be

$$u(x) = \frac{0.4}{1 \ + 10 \ e^{-2000(x-0.9)}} + 0.4,$$

which is a smooth approximation to the step function with discontinuity at $x = 0.9$ that jumps from 0.4 to 0.8. We assume workers have a linear utility function

$$v(x) = F + 1 - x.$$

Observe that the firm utility function u is not part of the family of utility functions studied in Lemma 3, since the separation between the two tiers occurs in the top quartile of workers. As a result, our example has the potential to have indeterminate behavior for a wider range of number of signals, far beyond just $S \leq 4$ signals. In fact, we show that this indeterminate behavior occurs for up to $S = 11$ signals.

As a baseline, in Table 1, we compute the number of matches, firm welfare, and worker welfare in the game without signals.[4] This game corresponds to firms sending offers to workers with score ≥ 0.98. Later, we will compare these values to the analogous welfare metrics in the game with signals.

[4] These computations follow from Lemma 6.

Table 1: Game with No Signals

Number of Matches	0.636
Firm Welfare	0.509
Worker Welfare	18.8

A.1 Equilibrium Analysis

First, we investigate the equilibrium structure of the game with signals for $1 \leq S \leq 50$ signals. We highlight that it is indeterminate whether only sending offers to signaling workers, i.e., $c_S = 1-\alpha/S$ and $c_N = 1$, is an equilibrium for $1 \leq S \leq 9$ signals.

We may compute from Lemma 4 the equilibrium structure given in Table 2. The three types of equilibria that we observe in this example are an equilibrium at $1 - \alpha/S$ (where firms only send offers to signaling workers), an equilibrium at 0.900 (where firms only send offers to top-tier workers), and a "middle equilibrium" with c_S between 0.6 and 0.9. In this discussion, we focus on the equilibrium at $c_S = 1 - \alpha/S$ for $1 \leq S \leq 9$ signals. This equilibrium is a corner equilibrium in which firms do not accept any non-signaling workers. The equilibrium at 0.900 and the middle equilibrium occur when the first-order condition is satisfied.

Table 2: Equilibrium Structure in Game with Signals

S	Equilibrium at $(1-\alpha/S)$?	Other Equilibrium < 0.9	Equilibrium ≥ 0.9
1	0.000	0.688	0.900
2	0.500	0.737	0.900
3	0.667	0.770	0.900
4	0.750	0.797	0.900
5	0.800	0.814	0.900
6	NO	N/A	0.900
7	NO	N/A	0.900
8	NO	N/A	0.900
9	NO	N/A	0.900
≥ 10	YES	N/A	N/A

The relevant factor that determines whether $c_S = 1-\alpha/S$ is an equilibrium for $1 \leq S \leq 9$ is how the expected amount of utility gained from sending an offer to a top-ranked non-signaling worker compares to the gain from sending an offer to a low-ranking signaling worker. In this regime, the lowest-ranked signaling worker is in the bottom tier, while the highest-ranked non-signaling worker in the top tier. Thus, it becomes a question of how the difference between the utilities derived from workers in the top tier and workers in the bottom tier compares to the difference in probability of acceptance between signaling and non-signaling workers. For $1 \leq S \leq 5$, the higher probability of acceptance dominates over the potential conversion from bottom tier to top tier. For $6 \leq S \leq 9$, the signal is diluted enough that the conversion from bottom tier to top tier dominates over increase in probability of acceptance.

The three types of equilibria demonstrate the two types of change that occur from varying the cutoff: change in utility and change in probability of acceptance. The equilibrium at $c_S = 1 - \alpha/S$ occurs when the increased probability of acceptance from signaling workers dominates over the utility gap between the two tiers. The equilibrium at 0.9 corresponds to each firm accepting only top tier workers from the set of signaling workers, due to the utility gap between the two tiers. The middle equilibrium occurs when the firms respond to signals exactly enough so that the probability of acceptance from a top tier non-signaling worker is reduced to the point where the first-order condition is satisfied.

The reason that a cutoff near 0.9 is an equilibrium for $1 \leq S \leq 9$ stems from the fact that it is the boundary between top tier and bottom tier workers. First, observe that no individual firm has a reason to make the cutoff much higher than 0.9, since all top tier workers are worth roughly the same utility. Second, since every other firm has not sent offers to any workers in their bottom tier, each individual firm has a reasonable probability of acceptance when sending offers to non-signaling workers in their top tier and thus does not want to lower its cutoff. For $S \geq 10$, the firm can fill its entire offer quota with signaling workers from the top tier, and since the probability of acceptance is higher among signaling workers than non-signaling workers, it does not make sense for the firm to waste offers on non-signaling workers. Therefore, when the number of available signals is large, the only equilibrium corresponds to sending offers to all of the signaling workers.

A.2 Welfare Metrics

We now consider how the number of matches, worker welfare, and firm welfare vary with respect to the number of signals S in this signaling market. We observe that the optimal number of signals with respect to any of the welfare metrics is between zero signals and eleven signals. However, the specific optimal value depends on the choice of welfare metric and choice of equilibrium. The main reason for this complication is that the welfare metrics face the following indeterminacies for small numbers of signals:

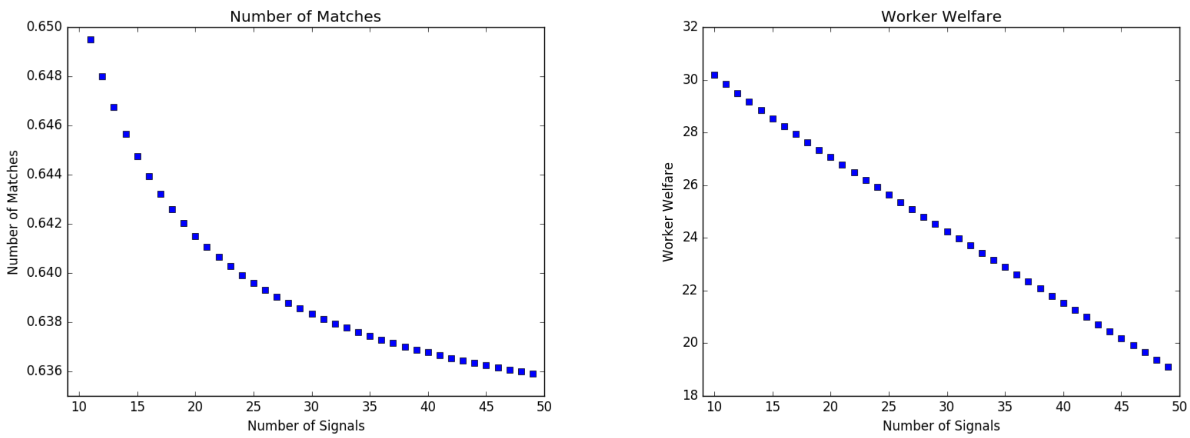
1. When there are multiple equilibria, the effect of varying signals on worker welfare, the number of matches, and firm welfare is indeterminate. This occurs for $1 \leq S \leq 5$ signals.
2. Even if we restrict our consideration to the firm-optimal equilibrium where firms only send offers to top-tier signaling workers, the effect of varying signals on worker welfare, the number of matches, and firm welfare is indeterminate for $S \geq 1$.
3. Even when the unique equilibrium is at the cutoff where firms only send offers to signaling workers, the effect of varying the number of signals on firm welfare can be indeterminate. One such example occurs at $10 \leq S \leq 12$, which demonstrates why it is necessary for Theorem 6 to have a greater threshold value than that of Theorem 4.
4. The comparison between firm welfare in the game with signals and the game with no signals is indeterminate, e.g., when there are $1 \leq S \leq 5$ signals and multiple equilibria.

Table 3: Number of Matches for $1 \leq S \leq 9$

S	Equilibrium at $(1 - \alpha/S)$?	Other Equilibrium < 0.9	Equilibrium ≈ 0.9
1	1.000	0.656	0.637
2	0.750	0.663	0.639
3	0.704	0.666	0.640
4	0.684	0.666	0.641
5	0.672	0.667	0.643
6	NO	N/A	0.644
7	NO	N/A	0.646
8	NO	N/A	0.648
9	NO	N/A	0.650

Table 4: Worker Welfare for $1 \leq S \leq 9$

S	Equilibrium at $(1 - \alpha/S)$?	Other Equilibrium < 0.9	Equilibrium ≈ 0.9
1	50.0	25.2	20.4
2	37.3	28.1	21.8
3	34.7	29.8	23.2
4	33.4	30.9	24.4
5	32.6	31.7	25.5
6	NO	N/A	26.6
7	NO	N/A	27.6
8	NO	N/A	28.5
9	NO	N/A	29.4



(a) Number of matches for $S \geq 10$

(b) Worker welfare for $S \geq 10$

Fig. 2: Effect of changing S on number of matches and worker welfare

Number of Matches and Worker Welfare We first study the number of matches and worker welfare, which behave similarly. When $1 \leq S \leq 9$, Table 3 shows the number of matches and Table 4 shows worker welfare. For $S \geq 10$, Figure 2a shows the number of matches and Figure 2b shows worker welfare.[5]

We observe that it is indeterminate how the number of matches and worker welfare change for $1 \leq S \leq 5$ signals. Due to the existence of multiple equilibria, the set of welfare metrics for any given number of signals has a wide range. As a result, the ranges spanned by these sets overlap across different numbers of signals, and the number of matches and worker welfare becomes highly dependent on choice of equilibrium.

For $S \geq 1$ signals, even if we restrict our consideration to the unique equilibrium where firms only send offers to top tier signaling workers, the direction of change in the number of matches and worker welfare is still indeterminate. For $S \leq 10$, where this equilibrium occurs exactly at the cutoff between top tier and bottom tier workers, the number of signaling workers who receive offers increases as a function of the number of signals. In this regime, the increase in quantity of offers to signaling workers overpowers the decrease in probability of acceptance, thus resulting in a greater number of matches and greater worker welfare. For $S \geq 10$, where the equilibrium occurs when firms only send offers to signaling workers, the increase in the number of signals causes a reduction

[5] These computations follow from Proposition 3 and Proposition 4.

in the probability of acceptance as well as a reduction in the average rank of firms who send offers to workers, which results in a decrease in the number of matches and worker welfare.

Table 5: Firm Welfare for $1 \leq S \leq 9$

S	Equilibrium at $(1 - 1/S)$?	Other Equilibrium < 0.9	Equilibrium ≈ 0.9
1	0.440	0.440	0.509
2	0.359	0.416	0.510
3	0.365	0.408	0.511
4	0.382	0.410	0.512
5	0.402	0.413	0.513
6	N/A	N/A	0.513
7	N/A	N/A	0.514
8	N/A	N/A	0.515
9	N/A	N/A	0.517

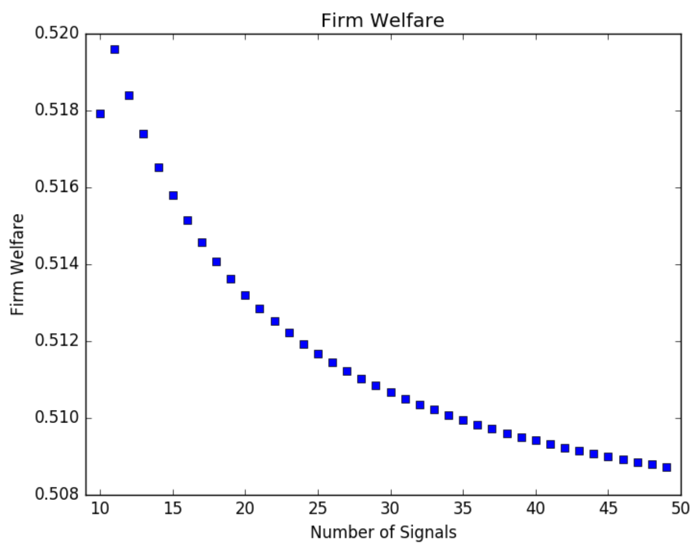


Fig. 3: Firm welfare for $S \geq 10$

Firm Welfare We now study firm welfare, whose behavior is more complicated. Table 5 shows firm welfare for $1 \leq S \leq 9$, and Figure 2a shows firm welfare for $S \geq 10$.[6] Like for the number of matches and worker welfare, we observe that it is indeterminate how firm welfare changes as the number of signals increases for small numbers of signals (i.e., $S \leq 9$ signals). Similarly, we observe that it is indeterminate how firm welfare changes as the number of signals increases for $1 \leq S \leq 5$. The existence of multiple equilibria similarly causes the possible ranges of firm welfare to overlap across different numbers of signals, so for small numbers of signals, firm welfare becomes highly dependent on the choice of equilibrium.

[6] These computations follow from Proposition 5.

For $S \geq 1$ signals, even if we restrict our consideration to the unique equilibrium where firms only send offers to top tier signaling workers, the direction of change in firm welfare is still indeterminate. For $S \leq 10$, the increase in quantity of offers to signaling workers overpowers the decrease in probability of acceptance. Thus firm welfare increases as a result of more top tier workers accepting offers. However, unlike for the number of matches and worker welfare, firm welfare also increases between $S = 10$ and $S = 11$. This occurs because the increase in the number of signaling workers continues to dominate over the reduction in probability of acceptance. As a result, the direction of change of firm welfare for certain smaller numbers of signals is indeterminate even when the game has a unique equilibrium at $c_S = 1 - \alpha/S$. This phenomenon demonstrates why it is necessary to allow Theorem 6 to have a greater threshold value than that of Theorem 4. For $S \geq 11$, firm welfare decreases because the reduction in probability of acceptance resulting from the increase in the number of signals becomes the dominant effect.

Unlike for worker welfare and the number of matches, the comparison between firm welfare in the game with signals and in the game without signals becomes indeterminate. In the set of equilibria where firms only send offers to top tier workers, signaling increases yield rate without any reduction to the utility derived from workers, thus resulting in a higher firm welfare in the game with signals. In the equilibria where the firms spend offers on bottom tier workers, the reduction in utility derived from the workers becomes the dominating effect and results in a lower firm welfare in the game with signals.

B Proofs for Section 3

B.1 Worker and Firm Strategies

We first prove Lemma 1.

Proof (Proof of Lemma 1). We show that cutoff strategies (or measure 0 deviations from them) are the only firm best-responses to a strategy profile in which the workers play anonymously and symmetrically and all other firms play anonymous strategies. Consider the decision that Firm 1 must make after receiving the workers' signals. If Firm 1 sends an offer to a signaling worker, then the symmetry of worker strategies and the anonymity of the other firms' strategies implies the probability that the probability this worker accepts the offer is the same as that of any other signaling worker. Let this probability be p. The same observation holds for non-signaling workers; let the probability that a non-signaling worker accepts an offer from Firm 1 be q.

These fixed probabilities imply Firm 1 should always send offers to the workers it values most among those who have and haven't signaled, since workers within a group will all accept with the same probability. Thus, in equilibrium, any action that Firm 1 plays with positive probability will always involve a cutoff for each group of workers, such that the firm will send an offer to a worker if and only if they are above the cutoff. It remains to show there exists a fixed cutoff that Firm 1 should always play, i.e., there is no mixing of strategies. Let's suppose that each firm receives a measure β of signals from workers. Suppose Firm 1 sends an f fraction of its offer quota to the workers who have signaled, with the remainder going to those who have not signaled. This means that the cutoff for signaling workers is $1 - f\gamma F/S$ and the cutoff for non-signaling workers is $1 - \frac{(1-f)\gamma F}{F-S}$. Its utility is

$$p\beta \int_{1-f\gamma F/S}^{1} u(x)\,dx + q(1-\beta) \int_{1-\frac{(1-f)\gamma F}{F-S}}^{1} u(x)\,dx.$$

This function is concave in f, since the second derivative with respect to f is

$$-\frac{p\beta\gamma^2F^2}{S^2}\cdot u'\left(1-\frac{f\gamma F}{S}\right)-\frac{q(1-\beta)\gamma^2F^2}{(F-S)^2}\cdot u'\left(\frac{(1-f)\gamma F}{F-S}\right)<0.$$

It follows that there is a unique optimum fraction f^* of Firm 1's offers that should be allocated to the workers who have signaled. Furthermore, by Jensen's inequality, any mixture involving suboptimal strategies does strictly worse. This shows that in any Nash equilibrium, a firm will always play the pure strategy of sending offers to a fixed fraction of the signaled workers.

We next prove Lemma 2.

Proof (Proof of Lemma 2). Notice that $c_S > c_N$ is not possible, because this would mean that signaling reduces the probability of receiving an offer. This scenario is not possible, because it would mean that no worker would want to signal, and so this could not arise.

Assume that $c_S < c_N$. Suppose that firm i applies the cutoff strategies to all but a set E_i of workers. We know that E_i has measure 0. Let E be the union of all of the sets E_i for $1 \le i \le F$. We know that E also has measure 0. Now, suppose that w is a worker not in E.

Since $\{w\}$ is a measure 0 set, regardless of the strategy of w, the set of worker strategies continues to be symmetric, and the firm strategy continues to be a cutoff strategy parameterized by c_S. Since $\gamma < 1/F$, we know that $c_S > 0$. This implies workers are not guaranteed an offer by signaling, and since $c_S < c_N$, signaling increases the chance of receiving an offer. Thus, each worker will signal to their top S firms.

B.2 Existence of Equilibria

In this section, we prove Theorem 1. In order to prove this result, we first show the following characterization of equilibria (Lemma 4), which itself relies on Propositions 1 and 2 (also proved in this subsection):

Lemma 4. *Suppose that firms are playing an equilibrium cutoff strategy where they accept all signaling workers with score $\ge c$. Then $c_S = 1-\gamma$ is never an equilibrium. For $c_S = 1-\frac{\gamma F}{S}$ to be an equilibrium, the following condition must be satisfied*

$$\frac{u(1)}{u\left(1-\frac{\gamma F}{S}\right)} \le \frac{1}{\gamma F}\left(\left(1-\frac{\gamma F}{S}\right)^{-S}-1\right).$$

For any $1-\frac{\gamma F}{S} < c_S < 1-\gamma$ to be an equilibrium, the following condition must be satisfied:

$$\frac{1-c^S}{F(1-c)}u(c) = Q_S(c)\frac{S}{F-S}\cdot u\left(\frac{F(1-\gamma)-cS}{F-S}\right).$$

For the remainder of the appendix, we drop the subscript on c_S. We also let

$$Q_S(c) = \frac{c^S}{F}\left(\sum_{k=0}^{F-S-1}\left(\frac{F(1-\gamma)-cS}{F-S}\right)^k\right) = \frac{c^S}{F\gamma-S+cS}\left(1-\frac{S}{F}\right)\left(1-\left(\frac{F(1-\gamma)-cS}{F-S}\right)^{F-S}\right).$$

Suppose the cutoff of every firm except Firm 1 is c, and suppose that Firm 1 chooses the optimal cutoff c' for itself. Observe that c is an equilibrium cutoff if and only if $c = c'$. We prove the following result about c', which is a core result in our proof of Lemma 4:

Proposition 1. *The following is the condition for playing a cutoff of* $c' = 1 - \frac{\gamma F}{S}$ *to be a best-response given that the other firms are each playing a cutoff of* c*:*

$$-\frac{1-c^S}{F(1-c)}u\left(1-\frac{\gamma F}{S}\right) + Q_S(c)\frac{S}{F-S}\cdot u(1) \leq 0$$

In this case, Firm 1's utility is an increasing function of c*.*

Furthermore, there exists a d *such that for* $c \in [1-\frac{\gamma F}{S}, d]$ *(which might be an empty interval), Firm 1's optimal cutoff is* $1-\gamma F/S$*, and for* $c \in [d, 1-\gamma]$ *, Firm 1's optimal cutoff is the unique value* $1-\frac{\gamma F}{S} < c' < 1-\gamma$ *satisfied by:*

$$-\frac{1-c^S}{F(1-c)}u(c') + Q_S(c)\frac{S}{F-S}\cdot u\left(\frac{F(1-\gamma)-c'S}{F-S}\right) = 0.$$

The following computation is useful in the proof of Proposition 1:

Proposition 2. *Let*

$$Y = \frac{c^S}{F\gamma - S + cS}\left(1-\left(\frac{F(1-\gamma)-cS}{F-S}\right)^{F-S}\right)$$

and

$$N = \frac{1-c^S}{S(1-c)}.$$

Then, $\frac{Y}{N}$ *is a decreasing function of* c *on the interval* $[1-\frac{\gamma F}{S}, 1]$*.*

Proof (Proof of Proposition 2). First we change variables to obtain that

$$Y = \frac{1-\left(1-\frac{f\gamma F}{S}\right)^S}{f\gamma F}$$

and

$$N = \frac{\left(1-\frac{f\gamma F}{S}\right)^S}{(1-f)\gamma F}\left(1-\left(1-\frac{(1-f)\gamma F}{F-S}\right)^{F-S}\right)$$

What we want to show is that $\frac{Y}{N}$ is an increasing function of f. It suffices to show that $\log Y - \log N$ is an increasing function of f. We see that

$$\log Y = -\log(f\gamma F) + \log\left(1-\left(1-\frac{f\gamma F}{S}\right)^S\right)$$

$$\log N = \log\left(\left(1-\frac{f\gamma F}{S}\right)^S\right) - \log((1-f)\gamma F) + \log\left(1-\left(1-\frac{(1-f)\gamma F}{F-S}\right)^{F-S}\right)$$

We see that

$$\frac{\partial \log Y}{\partial f} = -\frac{1}{f} + \frac{\gamma F\left(1-\frac{f\gamma F}{S}\right)^{S-1}}{1-\left(1-\frac{f\gamma F}{S}\right)^S}$$

$$\frac{\partial \log N}{\partial f} = -\frac{\gamma F\left(1-\frac{\gamma F}{S}\right)^{S-1}}{\left(1-\frac{\gamma F}{S}\right)^S} + \frac{1}{1-f} - \frac{\gamma F\left(1-\frac{(1-f)\gamma F}{F-S}\right)^{F-S-1}}{1-\left(1-\frac{(1-f)\gamma F}{F-S}\right)^{F-S}}$$

This implies that the derivative of $\log Y - \log N$ is:

$$-\frac{1}{f(1-f)} + \frac{\gamma F\left(1-\frac{\gamma F}{S}\right)^{S-1}}{\left(1-\left(1-\frac{f\gamma F}{S}\right)^{S}\right)\left(1-\frac{\gamma F}{S}\right)^{S}} + \frac{\gamma F\left(1-\frac{(1-f)\gamma F}{F-S}\right)^{F-S-1}}{1-\left(1-\frac{(1-f)\gamma F}{F-S}\right)^{F-S}}$$

We use the following three facts:

$$\frac{1}{(1-\frac{f\gamma F}{S})^{S})} \geq e^{f\gamma F}$$

$$\frac{\left(1-\frac{(1-f)\gamma F}{F-S}\right)^{F-S-1}}{\left(1-\left(1-\frac{(1-f)\gamma F}{F-S}\right)^{F-S}\right)} \geq \frac{e^{-(1-f)\gamma F}}{1-e^{-(1-f)\gamma F}}$$

$$\frac{\left(1-\frac{f\gamma F}{S}\right)^{S-1}}{\left(1-\left(1-\frac{f\gamma F}{S}\right)^{S}\right)} \geq \frac{e^{-f\gamma F}}{1-e^{-f\gamma F}}$$

It follows from these three facts that our expression is lower bounded by

$$\left(\frac{\gamma F}{1-e^{-f\gamma F}} - \frac{1}{f}\right) + \left(\frac{\gamma F e^{-(1-f)\gamma F}}{1-e^{-(1-f)\gamma F}} - \frac{1}{1-f}\right).$$

We substitute in $c = \gamma F$. Now, the expression becomes

$$\left(\frac{c}{1-e^{-fc}} - \frac{1}{f}\right) + \left(\frac{c e^{-(1-f)c}}{1-e^{-(1-f)c}} - \frac{1}{1-f}\right)$$

which is always positive for any $c \geq 0$.

Now, we prove Proposition 1.

Proof (Proof of Proposition 1). When each worker is allowed to signal to S firms where $S < F$, the cutoff must lie between $1 - \frac{\gamma F}{S}$ (since at most a measure γ set of workers can be offered to) and $1-\gamma$ (since the cutoff for signaling workers cannot be higher than the cutoff for non-signaling workers). In this case, Firm 1 sends out offers to a measure $(1-c')S/F$ set of signaling workers, and the cutoff for the non-signaling workers thus becomes $\frac{F(1-\gamma)-c'S}{F-S}$. The new welfare for Firm 1 is

$$U_S(c';c) = \frac{1-c^S}{F(1-c)}\int_{c'}^{1} u(x)\,dx + Q_S(c)\int_{\frac{F(1-\gamma)-c'S}{F-S}}^{1} u(x)\,dx.$$

Taking a derivative with respect to c', we get that the first-order condition is

$$-\frac{1-c^S}{F(1-c)}u(c') + Q_S(c)\frac{S}{F-S}\cdot u\left(\frac{F(1-\gamma)-c'S}{F-S}\right) = 0.$$

The second derivative is then

$$-\frac{1-c^S}{F(1-c)}u'(c') - \frac{S}{F-S}Q_S(c)\frac{S}{F-S}\cdot u'\left(\frac{F(1-\gamma)-c'S}{F-S}\right) < 0,$$

so utility is concave as a function of c', and there is at most 1 local maximum for the utility.

The condition for $c' = 1 - \frac{\gamma F}{S}$ to be the optimal cutoff for Firm 1 is:

$$-\frac{1-c^S}{F(1-c)} u\left(1 - \frac{\gamma F}{S}\right) + Q_S(c) \frac{S}{F-S} \cdot u(1) \leq 0. \tag{1}$$

We first show that $c' = 1 - \frac{\gamma F}{S}$ is Firm 1's optimal cutoff on some (possibly empty) interval $c \in [1 - 1/S, d]$ and satisfies $c' > 1 - \frac{\gamma F}{S}$ for all $c > d$. This is equivalent to showing that (1) occurs exactly on a (possibly empty) interval $c \in [1 - 1/S, d]$. Let $Y = \frac{c^S}{\gamma F - S + cS}\left(1 - \left(\frac{F(1-\gamma) - cS}{F-S}\right)^{F-S}\right)$ and $N = \frac{1-c^S}{S(1-c)}$. The above condition is equivalent to $-\frac{Y}{N} u\left(1 - \frac{\gamma F}{S}\right) + u(1) < 0$. It suffices to show that $\frac{Y}{N}$ is a decreasing function of c, and this follows from Proposition 2.

Second, observe that the welfare for Firm 1 when $c' = 1 - \frac{\gamma F}{S}$ is $\frac{1-c^S}{F(1-c)} \int_{1-\frac{\gamma F}{S}}^{1} u(x)\, dx$, which is an increasing function of c.

The condition for $c' = 1 - \gamma$ to be the optimal cutoff for Firm 1 is:

$$-\frac{1-c^S}{F(1-c)} u(1-\gamma) + Q_S(c) \frac{S}{F-S} \cdot u\left(1-\gamma\right) \geq 0.$$

We see that $\frac{F(Q_S(c))}{F-S}$ is the fraction of workers who don't signal to Firm 1 that accept Firm 1 given an offer. We see that $\frac{1-c^S}{S(1-c)}$ is the fraction of workers who signal to Firm 1 that accept Firm 1 given an offer. We know that the second expression must be larger than the first expression, which implies that this expression is always negative. Thus, $c' = 1-\gamma$ is never the optimal cutoff for Firm 1.

If $c' = 1 - \frac{\gamma F}{S}$, it follows from the concavity of Firm 1's welfare in c' that there are no interior solutions to the first-order condition. If $c' \neq 1 - \frac{\gamma F}{S}$, it follows from the concavity of Firm 1's welfare in c' coupled with the fact that $c' \neq 1 - \gamma$ that there is exactly one interior solution to the first-order condition.

Proposition 1 is the key ingredient in our proof of Lemma 4. Now, we are ready to prove Lemma 4.

Proof (Proof of Lemma 4). The following results follow from Proposition 1: It follows that $c = 1 - \gamma$ is not an equilibrium. It also follows that it is indeterminate whether or not $c = 1 - \frac{\gamma F}{S}$ is an equilibrium, and the condition is:

$$\frac{u(1)}{u\left(1 - \frac{\gamma F}{S}\right)} < \frac{1 - \left(1 - \frac{\gamma F}{S}\right)^S}{\gamma F \left(1 - \frac{\gamma F}{S}\right)^S} = \frac{1}{\gamma F}\left(\left(1 - \frac{\gamma F}{S}\right)^{-S} - 1\right).$$

The condition for $1 - \frac{\gamma F}{S} < c < 1 - \gamma$ to be an equilibrium is:

$$\frac{1-c^S}{F(1-c)} u(c) = Q_S(c) \frac{S}{F-S} \cdot u\left(\frac{F(1-\gamma) - cS}{F-S}\right).$$

The existence of an equilibrium also follows from Proposition 1:

Proof (Proof of Theorem 1). If $c = 1 - \frac{\gamma F}{S}$ is an equilibrium, we have proved the statement. Suppose that $c = 1 - \frac{\gamma F}{S}$ is not an equilibrium. Then, it follows from Proposition 1 that for each

$1 - \frac{\gamma F}{S} \le c \le 1 - \gamma$, the value c' is in the interval $(1 - \frac{\gamma F}{S}, 1 - \gamma)$ and is uniquely defined by the condition:

$$-\frac{1 - c^S}{F(1-c)} u(c') + Q_S(c) \frac{S}{F-S} \cdot u\left(\frac{F(1-\gamma) - c'S}{F-S}\right) = 0.$$

Notice that c' is a continuous function in c, which implies that the function $D(c) = c'(c) - c$ is a continuous function in c. Notice that $D(1 - \frac{\gamma F}{S}) > 0$ and $D(1 - \frac{1}{F}) < 0$. By the intermediate value theorem, this implies that there exists a value $c \in (1 - \frac{\gamma F}{S}, 1 - \gamma)$ such that $c'(c) = c$, which is an equilibrium.

B.3 Reverse Implication for Equilibria

Now we show for any symmetric, anonymous equilibrium, if workers signal to their top S firms, then the case where $c_S = c_N$, i.e., where signaling does not change the probability of receiving an offer, is never an equilibrium.

Lemma 5. *Consider a symmetric, anonymous equilibria where workers signal to their top S firms. Then it necessarily holds that $c_S < c_N$.*

Proof (Proof of Lemma 5). If $c_S = c_N$, then we know that $c_S = 1 - \gamma$, which we know is never an equilibrium by Proposition 1.

It follows from Lemma 5, coupled with Lemma 2, that the symmetric, anonymous equilibria in which firms respond to signals (i.e., $c_S < c_N$) are exactly the set of symmetric, anonymous equilibria where workers always signal to their S most preferred firms.

C Proofs for Section 4.1

C.1 The Game with No Signals

We compute firm welfare, worker welfare, and the number of matched workers in the market with no signals. These quantities will be useful in later proofs.

Lemma 6. *Suppose that there are no signals. Then, (a) the number of matched workers is $1 - (1-\gamma)^F$, (b) the total worker welfare is $\gamma \sum_{i=1}^{F} (1-\gamma)^{i-1} v(i)$, and (c) the total firm welfare is*

$$\frac{(1 - (1-\gamma)^F)}{\gamma} \int_{1-\gamma}^{1} u(x)\, dx.$$

Proof. In the market with no signals, firms propose to the γ measure set of workers with a score of at least $1-\gamma$. We first count the number of unmatched workers. Observe that a worker is unmatched if their score is below $1-\gamma$ for every single firm. This happens to a $(1-\gamma)^F$ fraction of workers, thus implying there are $1 - (1-\gamma)^F$ matched workers.

To compute firm welfare, we first consider the welfare of a single firm and then multiply this value by F. Due to the independence between worker score and firm orderings, we know that the firm welfare is $\int_{1-\gamma}^{1} u(x)dx$ times the fraction of workers who accept the offer conditional on getting an offer. Observe that the probability that the firm is ranked i is $\frac{1}{F}$. A $(1-\gamma)^{i-1}$ fraction of workers do not get offers from their top $i-1$ firms. Thus the fraction of workers who would accept this firm given an offer is $\frac{1}{F}\sum_{i=1}^{F}(1-\gamma)^{i-1} = \frac{1-(1-\gamma)^F}{F\gamma}$ as desired.

To compute worker welfare, we observe that $(1-\gamma)^{i-1}$ fraction of workers do not get offers from their top $i-1$ firms, and a γ fraction of these workers get an offer from their i-th ranked firm. This means that $\gamma(1-\gamma)^{i-1}$ get and accept an offer from their i-th highest ranked firm. Summing over $1 \le i \le F$ gives us the desired total worker welfare.

C.2 Welfare and Number of Matches

In this subsection, we prove Theorem 2 and Theorem 3. For these proofs, a critical ingredient is to consider how the value of the cutoff affects the number of matches, firm welfare, and worker welfare.

First, we consider the number of matches.

Proposition 3. *The number of matches is* $1 - c^S \left(\frac{F(1-\gamma)-cS}{F-S}\right)^{F-S}$. *This quantity is a decreasing function of c, and a nonzero number of signals always produces more matches than no signals.*

Proof (Proof of Proposition 3). The number of unmatched workers is the fraction of workers who did not meet the cutoff amongst firms who they signaled to or to firms who they did not signal to. This is $c^S \left(\frac{F(1-\gamma)-cS}{F-S}\right)^{F-S}$ workers.

Here, we plug in the $f = \frac{S(1-c)}{\gamma F}$ to obtain $\left(1 - \frac{f\gamma F}{S}\right)^S \left(1 - \frac{(1-f)\gamma F}{F-S}\right)^{F-S}$. We take the log of this to obtain

$$S \log\left(1 - \frac{f\gamma F}{S}\right) + (F-S)\log\left(1 - \frac{(1-f)\gamma F}{F-S}\right).$$

Observe that the derivative is $-\frac{\gamma F}{1-\frac{f\gamma F}{S}} + \frac{\gamma F}{1-\frac{(1-f)\gamma F}{F-S}} < 0$ since $1 - \frac{f\gamma F}{S} \leq 1 - \frac{(1-f)\gamma F}{F-S}$. Thus, the number of unmatched workers decreases as f increases, which means that the number of matches decreases as the cutoff increases. Moreover, this means the minimum number of matches is obtained at the maximum cutoff $c = 1-\gamma$ (which corresponds to no signaling), so a nonzero number of signals always produces more matches than no signals, since $c = 1-\gamma$ is never an equilibrium by Theorem 1

Now, we consider worker welfare.

Proposition 4. *The worker welfare*

$$(1-c)\sum_{i=1}^{S} c^{i-1} v(i) + c^S \left(\frac{F\gamma - S + cS}{F-S}\right)\left(\sum_{i=S+1}^{F}\left(\frac{F(1-\gamma)-cS}{F-S}\right)^{i-S-1} v(i)\right)$$

is a decreasing function of c, and a nonzero number of signals always produces more matches than no signals.

Proof (Proof of Proposition 4). Suppose that a worker has signaled to a firm. For $i \leq S$ fraction of workers who get rejected from their top $i-1$ choices is c^{i-1}, and so the fraction of workers who get and accept a offer from their i-th best firm is $(1-c)c^{i-1}$. We see that a c^S fraction of workers who do not get an offer from any firm whom they signaled to. Similar logic shows that the fraction of workers who get and accept an offer from the i-th best firm for $i \geq S+1$ is $c^S \left(\frac{\gamma F\gamma - S + cS}{F-S}\right)\left(\frac{F(1-\gamma)-cS}{F-S}\right)^{i-S-1}$. This implies that the total worker welfare is:

$$(1-c)\sum_{i=1}^{S} c^{i-1} v(i) + c^S \left(\frac{F\gamma - S + cS}{F-S}\right)\left(\sum_{i=S+1}^{F}\left(\frac{F(1-\gamma)-cS}{F-S}\right)^{i-S-1} v(i)\right).$$

We write this with the substitution $f = \frac{S(1-c)}{\gamma F}$ to obtain

$$\frac{f\gamma F}{S}\sum_{i=1}^{S}\left(1 - \frac{f\gamma F}{S}\right)^{i-1} v(i) + \left(1 - \frac{f\gamma F}{S}\right)^S \left(\frac{(1-f)\gamma F}{F-S}\right)\left(\sum_{i=S+1}^{F}\left(1 - \frac{(1-f)\gamma F}{F-S}\right)^{i-S-1} v(i)\right).$$

First, we show that the derivatives $\frac{f\gamma F}{S}\left(1-\frac{f\gamma F}{S}\right)^{i-1}$ are positive. We see this by taking a log to obtain $\log f+(i-1)\log(1-\frac{f\gamma F}{S})$ and then a derivative to obtain $\frac{1}{f}-\frac{i-1}{S}\frac{\gamma F}{1-\frac{f\gamma F}{S}}\geq\frac{1}{f}-\frac{S-1}{S}\frac{\gamma F}{1-\frac{f\gamma F}{S}}$. This is equivalent to showing that $S-f\gamma F\geq Sf\gamma F-f\gamma F$, which follows from the fact that $f\leq 1$ and $\gamma F\leq 1$.

Now, we show that the derivatives $\left(1-\frac{f\gamma F}{S}\right)^S\left(\frac{(1-f)\gamma F}{F-S}\right)\left(1-\frac{(1-f)\gamma F}{F-S}\right)^{i-S-1}$ are negative. We see this by taking the log to obtain $S\log(1-\frac{f\gamma F}{S})+\log(1-f)+j\log(1-\frac{(1-f)\gamma F}{F-S})$ where $0\leq j\leq F-S-1$. We see that the derivative is $-\frac{\gamma F}{1-\frac{f\gamma F}{S}}-\frac{1}{1-f}+\frac{j}{F-S}\frac{\gamma F}{1-\frac{(1-f)\gamma F}{F-S}}\leq-\frac{\gamma F}{1-\frac{f\gamma F}{S}}+\frac{\gamma F}{1-\frac{(1-f)\gamma F}{F-S}}\leq 0$.

Thus, the minimum derivative is obtained when v is a constant function. In this case, the worker welfare is proportional to the number of matches, and we already showed that the number of matches is an increasing function of f. Thus, worker welfare decreases as c increases. Moreover, this means the minimum number of matches is obtained at the maximum cutoff $c=1-\gamma$ (which corresponds to no signaling), so a nonzero number of signals always produces more matches than no signals, since $c=1-\gamma$ is never an equilibrium by Theorem 1.

Now, we consider firm welfare.

Proposition 5. *The firm welfare*

$$\frac{1-c^S}{(1-c)}\int_c^1 u(x)\,dx+\frac{c^S(F-S)}{F\gamma-S+cS}\left(1-\left(\frac{F(1-\gamma)-cS}{F-S}\right)^{F-S}\right)\int_{\frac{F(1-\gamma)-cS}{F-S}}^1 u(x)\,dx$$

is an increasing function of c.

A key ingredient in the proof of this proposition is the following computation.

Proposition 6. *The function $Q_S(c)$ is an increasing function of c in the interval* $[1-\frac{\gamma F}{S},1-\gamma]$.

Proof (Proof of Proposition 6). To show that $Q_S(c)$ is increasing with respect to c, it suffices to show that each term of the expanded sum is increasing with respect to c. To make our computation easier, we substitute $f=\frac{S(1-c)}{F\gamma}$. Note that we want the derivative of $Q_S(c)$ with respect to f to be *decreasing*. We then have, since $k<F-S$,

$$\frac{\partial}{\partial f}\left(\frac{(1-\frac{f\gamma F}{S})^S}{F}\left(1-\frac{\gamma F(1-f)}{F-S}\right)^k\right)=\left(\frac{(1-\frac{f\gamma F}{S})^S}{F}\left(1-\frac{\gamma F(1-f)}{F-S}\right)^k\right)\left(-\frac{\gamma F}{1-\frac{f\gamma F}{S}}+\frac{k\gamma F}{F-S}\frac{1}{1-\frac{\gamma F(1-f)}{F-S}}\right)$$
$$<\gamma F\left(\frac{(1-\frac{f\gamma F}{S})^S}{F}\left(1-\frac{\gamma F(1-f)}{F-S}\right)^k\right)\left(-\frac{1}{1-\frac{f\gamma F}{S}}+\frac{1}{1-\frac{\gamma F(1-f)}{F-S}}\right).$$

It follows that the left-hand side is negative if $1-\frac{f\gamma F}{S}\leq 1-\frac{\gamma F(1-f)}{F-S}$, which always holds.

We now prove Proposition 5:

Proof (Proof of Proposition 5). Notice that the total firm welfare is F times the welfare of Firm 1, and so that gives the desired formula. Furthermore, for a fixed number of firms, the firm welfare is proportional to Firm 1's welfare, so we consider the impact of changing c on Firm 1. It is not clear whether this result is true when c is not an equilibrium, so instead we use the envelope theorem to circumvent this problem. We use the notation from Proposition 1 and let c' be the solution to the first-order condition for $c\in[d,1-\gamma]$. By the envelope theorem, we have that

$$\frac{\partial U}{\partial c}(c';c)=\frac{1}{F}\frac{d}{dc}\left(\frac{1-c^S}{1-c}\right)\int_{c'}^1 u(x)\,dx+Q'_S(c)\int_{\frac{F(1-\gamma)-c'S}{F-S}}^1 u(x)\,dx.$$

We see that $U(c';c)$ is monotonically increasing because $Q_S(c)$ is increasing by Proposition 6 and $\frac{1-c^S}{1-c}$ is also increasing. In particular, if $c_1 < c_2$ are both interior equilibria, then firms prefer c_2.

Now, let's consider how interior equilibria compare to the potential equilibrium at $1 - \frac{\gamma F}{S}$. Suppose that the cutoff $1 - \frac{\gamma F}{S}$ is an equilibrium. When $c = d$, the first-order condition is satisfied at $c' = 1 - \frac{\gamma F}{S}$. By Proposition 1, we know that utility is an increasing function of c for $c \leq d$ (where $c' = 1 - \frac{\gamma F}{S}$). This means that the equilibrium at $c' = 1 - \frac{\gamma F}{S}$ is strictly worse for the firms than any other equilibria.

Now, we prove Theorem 2 and Theorem 3.

Proof (Proof of Theorem 2). This follows from Proposition 3 (for the number of matches), Proposition 5 (for firm welfare), and Proposition 4 (for worker welfare).

Proof (Proof of Theorem 3). This follows from Lemma 6 (for the game with no signals), coupled with Proposition 3 (for the number of matches) and Proposition 4 (for worker welfare).

D Proofs for Section 4.2

D.1 One Equilibrium in the Limit

In this subsection, we prove Theorem 4. The following proposition will be useful in the proof (and will be used in the next subsection as well).

Proposition 7. *For any nonzero number of signals S, there is a unique equilibrium at $f = 1$ for any number of firms $F > S$ when for $S \leq \frac{F}{2}$:*

$$\frac{1 - e^{-f\gamma F}}{f\gamma F} u\left(1 - \frac{f\gamma F}{S}\right) - \frac{e^{-f\gamma F}}{(1-f)\gamma F}\left(1 - \left(1 - \frac{(1-f)\gamma F}{S}\right)^S\right) u(1) > 0$$

and for $S > F/2$:

$$\frac{1 - e^{-f\gamma F}}{f\gamma F} u(1 - \frac{f\gamma F}{S}) - e^{-f} u(1) > 0.$$

Proof. First, we reparameterize the first-order condition for multiple signals. Let $f = \frac{S(1-c)}{\gamma F}$ denote the fraction of workers proposed to by the firm who also signaled to the firm. Notice that $\frac{S}{F} \leq f \leq 1$. We know that $f = \frac{S}{F}$ is never an equilibrium (this follows from the fact that $c = 1 - \gamma$ is not an equilibrium). Notice that the condition for $\frac{S}{F} < f < 1$ to be an equilibrium is:

$$\frac{1 - \left(1 - \frac{f\gamma F}{S}\right)^S}{f\gamma F} u\left(1 - \frac{f\gamma F}{S}\right) - \frac{\left(1 - \frac{f\gamma F}{S}\right)^S}{(1-f)\gamma F}\left(1 - \left(1 - \frac{(1-f)\gamma F}{F-S}\right)^{F-S}\right) u\left(1 - \frac{(1-f)\gamma F}{F-S}\right) = 0.$$

Here, $f = 1$ and $f = 0$ are just defined as whatever they are in the limit. The condition for $f = 1$ to be an equilibrium is that the LHS is positive. Thus, for $f = 1$ to be the only equilibrium, it suffices to have that the LHS is always positive. We want to show that the LHS is *uniformly* bounded on $f \in [S/F, 1]$. We consider two cases: $S \leq F/2$ and $S > F/2$. In both cases, we want to eliminate the F from our condition and make this an increasing function of S so that we can apply Dini's theorem later.

For the first case, observe that this is lower bounded by the following expression:

$$\frac{1 - e^{-f\gamma F}}{f\gamma F} u \left(1 - \frac{f\gamma F}{S}\right) - \frac{e^{-f\gamma F}}{(1-f)\gamma F} \left(1 - \left(1 - \frac{(1-f)\gamma F}{S}\right)^S\right) u(1)$$

For the second case, observe that this is lower bounded by the following expression:

$$\frac{1 - e^{-f\gamma F}}{f\gamma F} u(1 - \frac{f\gamma F}{S}) - \frac{e^{-f\gamma F}}{(1-f)\gamma F}(1-f)\gamma F u(1).$$

Now, we prove Theorem 4.

Proof (Proof of Theorem 4). Our main ingredients are Dini's theorem and Proposition 7. As given in the theorem statement, we assume that $\alpha = \gamma F$ is a constant for $\alpha < 1$.

Observe that the LHS of the first expression $\frac{1-e^{-f\alpha}}{\alpha f} u(1 - \frac{f\alpha}{S}) - \frac{e^{-f\alpha}}{(1-f)\alpha} \left(1 - \left(1 - \frac{(1-f)\alpha}{S}\right)^S\right) u(1)$ in the proposition statement approaches:

$$u(1) \left(\frac{1 - e^{-f\alpha}}{f\alpha} - \frac{e^{-f\alpha}(1 - e^{-(1-f)\alpha})}{(1-f)\alpha}\right).$$

Since the functions $\frac{1-e^{-f\alpha}}{f\alpha} u(1 - \frac{f\alpha}{S}) - \frac{e^{-f\alpha}}{(1-f)\alpha} \left(1 - \left(1 - \frac{(1-f)\alpha}{S}\right)^S\right) u(1)$ are continuous, defined on a compact interval $[0, 1]$, monotonically increasing, and approach a continuous function in the limit, we know that the convergence is uniform by Dini's theorem. For a fixed α, for $0 \leq f \leq 1$, this limiting function is uniformly lower bounded by a constant multiple of $u(1)$.

Observe that the LHS of the second expression $\frac{1-e^{-f\alpha}}{f\alpha} u(1 - \frac{f\alpha}{S}) - e^{-f\alpha} u(1)$ in the proposition statement approaches:

$$u(1) \left(\frac{1 - e^{-f\alpha}}{f\alpha} - e^{-f\alpha}\right).$$

Since the functions $\frac{1-e^{-f\alpha}}{f\alpha} u(1 - \frac{f\alpha}{S}) - e^{-f\alpha} u(1)$ are continuous, defined on the compact interval $[0.5, 1]$, monotonically increasing, and approach a continuous function in the limit, we know that the convergence is uniform by Dini's theorem. For any fixed $0 < \alpha < 1$, for $0.5 \leq f \leq 1$, this limiting function is uniformly lower bounded by a constant multiple of $u(1)$.

D.2 Understanding the Limiting Equilibrium

In this subsection, we prove Theorem 5 and Lemma 3. For these proofs, we need to consider the boundary equilibrium $c = 1 - \frac{\alpha}{S}$, where firms only send offers to those workers who have signaled to them. We consider a large enough number of signals so that this is the unique equilibrium as given in Theorem 4.

We now prove Theorem 5, which analyzes how the number of unmatched workers and the worker welfare in this equilibrium change with the number of signals.

Proof (Proof of Theorem 5). By Theorem 4, we know that there exists C such that for all $S \geq C$, the equilibrium at $c = 1 - \frac{\alpha}{S}$ is the unique equilibrium.

The number of unmatched workers is $\left(1 - \frac{\alpha}{S}\right)^S$ for $S \geq C$. This increases as a function of S. Thus, the number of matches decreases as a function of the number of signals for any nonzero number of signals. Since the number of unmatched workers is equal to this expression at $S = F$, we know any number of signals $S \geq C$ beats no signals.

At the minimum equilibrium, we see that worker welfare is

$$\frac{\alpha}{S}\sum_{i=1}^{S}\left(1-\frac{\alpha}{S}\right)^{i-1}v(i).$$

By Lemma 6, the worker welfare with no signals is

$$\frac{\alpha}{F}\sum_{i=1}^{F}\left(1-\frac{\alpha}{F}\right)^{i-1}v(i).$$

The finite difference between S signals and $S+1$ signals is:

$$\frac{\alpha}{S+1}\left(1-\frac{\alpha}{S+1}\right)^{S}v(S+1)-\sum_{i=1}^{S}v(i)\left(\frac{\alpha}{S}\left(1-\frac{\alpha}{S}\right)^{i-1}-\frac{\alpha}{S+1}\left(1-\frac{\alpha}{S+1}\right)^{i-1}\right).$$

Notice that this expression is maximized when $v(S+1)=v(S)$. Furthermore, this expression is maximized when $v(1)=v(2)=\ldots=v(S)$. Thus, it suffices to show that the worker welfare is a decreasing function of S for a constant utility function v. However, worker welfare for a constant utility function is proportional to the number of matches, which we have already shown is a decreasing function of the number of signals. Furthermore, since the case of $S=F$ in the formula for worker welfare is equivalent to worker welfare with no signals, we see that $S\geq C$ signals always beats no signal.

We prove Theorem 6.

Proof (Proof of Theorem 6). The firm welfare is simply

$$\frac{S}{\alpha}\left(1-\left(1-\frac{\alpha}{S}\right)^{S}\right)\int_{1-\alpha/S}^{1}u(x)\,dx.$$

By Lemma 6, notice that firm welfare in the market with no signals is equal to this function evaluated at $S=F$.

Let $f(S)=1-(1-\frac{\alpha}{S})^{S}$. We take a derivative with respect to S, and multiply by S^2 to obtain:

$$f(S)\left(-Su(1-\frac{\alpha}{S})+\frac{S^2}{\alpha}\int_{1-\alpha/S}^{1}u(x)\,dx\right)+f'(S)S^2\frac{S}{\alpha}\int_{1-\alpha/S}^{1}u(x)\,dx.$$

The first term is always positive, since $u(x)>u(1-\alpha/S)$ for $x>1-\alpha/S$, and the second term is always negative. Therefore, the sign of this is indeterminate in general. Nonetheless, we show that this expression becomes determinate in the limit as long as $u'(1)\neq\frac{u(1)}{e-1}$.

We multiply by S^2 to obtain

$$f(S)\left(-Su(1-\frac{\alpha}{S})+\frac{S^2}{\alpha}\int_{1-\alpha/S}^{1}u(x)\,dx\right)+f'(S)S^2\frac{S}{\alpha}\int_{1-\alpha/S}^{1}u(x)\,dx.$$

Observe that $f(S)\to 1-e^{-\alpha}$ and $f'(S)S^2\to -0.5e^{-\alpha}\alpha^2$. Observe that $\frac{S}{\alpha}\int_{1-\alpha/S}^{1}u(x)\,dx\to u(1)$, so we can't directly take a limit. Observe that a change of variables from $-Su(1-\frac{\alpha}{S})+\frac{S^2}{\alpha}\int_{1-\alpha/S}^{1}u(x)\,dx$ gives $\frac{-hu(1-\alpha h)+\frac{1}{\alpha}\int_{1-\alpha h}^{1}u(x)\,dx}{h^2}$. Applying L'Hopital, we obtain $\frac{u(1-\alpha h)-u(1-\alpha h)+h\alpha u'(1-\alpha h)}{2h}$. This approaches $0.5\alpha u'(1)$.

Thus the decreasing condition becomes $0.5\alpha u'(1)(1-e^{-\alpha}) < 0.5e^{-\alpha}\alpha^2 u(1)$. This simplifies to $u'(1) < \frac{\alpha u(1)}{e^{\alpha}-1}$. The increasing condition similarly becomes $u'(1) > \frac{\alpha u(1)}{e^{\alpha}-1}$.

Thus, we've show that if $u'(1) < \frac{\alpha u(1)}{e^{\alpha}-1}$, then there exists a threshold value C such that if $S \geq C$, then the firm welfare is decreasing as a function of S. Notice that a corollary of this result is that for $F > C$, we have that no signals is worse than any number of signals $S \geq C$. Moreover, we've show that if $u'(1) > \frac{\alpha u(1)}{e^{\alpha}-1}$, then there exists a threshold value C such that if $S \geq C$, then the firm welfare is increasing as a function of S. Notice that a corollary of this result is that for $F > C$, we have that no signals is better than any number of signals $S \geq C$.

We now prove Lemma 3 (an explicit condition that guarantees the limiting result occurs for $S \geq 4$ signals.) The following proposition will be useful in the proof.

Lemma 7. *Suppose that the utility function u satisfies:*

$$\frac{u(1)}{u(0.75)} \leq 1.2.$$

Then, for any fixed $\alpha \geq 0.8$ and for any $S \leq 4$ and any $F > S$, the cutoff $c = 1 - \frac{\alpha}{S}$ is the unique equilibrium in the market with S signals and F firms with utility function u.

Proof (Proof of Lemma 7). By Proposition 7, it suffices to show that for $0 \leq f \leq 1$:

$$\frac{1-e^{-f\alpha}}{f\alpha e^{-f\alpha}} \frac{1}{E(f)} > \frac{u(1)}{u(1-\frac{f\alpha}{S})}$$

where $E(f) = \frac{1-\left(1-\frac{(1-f)\alpha}{S}\right)^S}{(1-f)\alpha}$ and for $0.5 \leq f \leq 1$:

$$\frac{1-e^{-f\alpha}}{f\alpha e^{-f\alpha}} > \frac{u(1)}{u(1-\frac{f\alpha}{S})}.$$

First, we observe that for $\alpha \geq 0.8$ and $0.5 \leq f \leq 1$, we have that

$$\frac{1-e^{-f\alpha}}{f\alpha e^{-f\alpha}} \geq 1.22.$$

For $0 \leq f \leq 0.5$, we have that this expression is lower bounded by 1. For $S \geq 4$, we also have that $E(f) \leq 1$ for $0.5 \leq f \leq 1$, and $E(f) \leq 0.8597$ for $0 \leq f \leq 0.5$.

Putting these facts together and using that $u(1-\frac{f\alpha}{S}) \geq u(1-\frac{1}{S}) \geq u(0.75)$, we obtain that $f = 1$ is the only equilibrium for $S \geq 4$ if

$$\frac{u(1)}{u(0.75)} \leq 1.16.$$

Now, we prove Lemma 3.

Proof (Proof of Lemma 3). Observe that $\frac{u(1)}{u(0.75)} \leq 1.16$ for utility functions in this family, which means that by Lemma 7, for each $S \geq 4$, there is a unique equilibrium at $c = 1 - \frac{\alpha}{S}$. In order to show that firm welfare is a decreasing function of S for $S \geq 4$, it suffices to show that for $S \geq 4$, the following expression is negative:

$$f(S)\left(-Su(1-\frac{\alpha}{S}) + \frac{S^2}{\alpha}\int_{1-\alpha/S}^{1} u(x)\,dx\right) + f'(S)S^2\frac{S}{\alpha}\int_{1-\alpha/S}^{1} u(x)\,dx.$$

We use the fact that for $S \geq 4$ and $\alpha \geq 0.8$, we know $0.143 \leq 0.5e^{-\alpha}\alpha^2 \leq -f'(S)S^2$ and $f(S) \leq 0.59$. For the expression to be negative, what we basically need is that

$$0.59\left(-u(1-\frac{\alpha}{S}) + \frac{S}{\alpha}\int_{1-\alpha/S}^{1} u(x)\,dx\right) \leq \frac{0.143}{\alpha}\int_{1-\alpha/S}^{1} u(x)\,dx.$$

This is equivalent to:

$$u(1-\frac{\alpha}{S}) \geq \frac{1}{\alpha}\left(\int_{1-\alpha/S}^{1} u(x)\,dx\right)(S-0.24).$$

A stronger version of this condition is that

$$u(1-\frac{1}{S}) \geq u(1)\frac{S-0.24}{S}$$

which can be rewritten as $u(y) \geq u(1)(0.76+0.24y)$ for $y \geq 0.75$ as desired.