

Simple Analysis of Sparse, Sign-Consistent JL

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Abstract

Allen-Zhu, Gelashvili, Micali, and Shavit construct a sparse, sign-consistent Johnson-Lindenstrauss distribution, and prove that this distribution yields an essentially optimal dimension for the correct choice of sparsity. However, their analysis of the upper bound on the dimension and sparsity requires a complicated combinatorial graph-based argument similar to Kane and Nelson’s analysis of sparse JL. We present a simple, combinatorics-free analysis of sparse, sign-consistent JL that yields the same dimension and sparsity upper bounds as the original analysis. Our analysis also yields dimension/sparsity tradeoffs, which were not previously known.

As with previous proofs in this area, our analysis is based on applying Markov’s inequality to the p th moment of an error term that can be expressed as a quadratic form of Rademacher variables. Interestingly, we show that, unlike in previous work in the area, the traditionally used Hanson-Wright bound is *not* strong enough to yield our desired result. Indeed, although the Hanson-Wright bound is known to be optimal for gaussian degree-2 chaos, it was already shown to be suboptimal for Rademachers. Surprisingly, we are able to show a simple moment bound for quadratic forms of Rademachers that is sufficiently tight to achieve our desired result, which given the ubiquity of moment and tail bounds in theoretical computer science, is likely to be of broader interest.

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1 Introduction

In many modern algorithms that process high dimensional data, it is beneficial to preprocess the data through a dimensionality reduction scheme that preserves the geometry of the data. Dimensionality reduction schemes have been applied in streaming algorithms [22] as well as algorithms for numerical linear algebra [29], feature hashing [27], graph sparsification [25], and many other areas. The geometry-preserving objective can be expressed mathematically as follows. The goal is to construct a probability distribution \mathcal{A} over $m \times n$ real matrices that satisfies the following condition for any $x \in \mathbb{R}^n$:

$$\mathbb{P}_{A \in \mathcal{A}}[(1 - \epsilon)\|x\|_2 \leq \|Ax\|_2 \leq (1 + \epsilon)\|x\|_2] > 1 - \delta. \quad (1)$$

An upper bound on the dimension m achievable by a probability distribution \mathcal{A} that satisfies (1) is given in the following lemma, which is a central result in the area of dimensionality reduction:

► **Lemma 1** (Johnson-Lindenstrauss [15]). *For any positive integer n and parameters $0 < \epsilon, \delta < 1$, there exists a probability distribution \mathcal{A} over $m \times n$ real matrices with $m = \Theta(\epsilon^{-2} \log(1/\delta))$ that satisfies (1).*

The optimality of the dimension m achieved by Lemma 1 was recently proven in [16, 14].

For many applications of dimensionality reduction schemes, it can be useful to consider probability distributions over sparse matrices in order to speed up the projection time. Here, sparsity refers to the constraint that there are a small number of nonzero entries in each column. In this context, Kane and Nelson [18] constructed a sparse JL distribution, improving the work of Achlioptas [1] and Dasgupta et al. [6], and proved the following:

► **Theorem 2** (Sparse JL [18]). *For any positive integer n and $0 < \epsilon, \delta < 1$, there exists a probability distribution \mathcal{A} over $m \times n$ real matrices with $m = \Theta(\epsilon^{-2} \log(1/\delta))$ and sparsity $s = \Theta(\epsilon^{-1} \log(1/\delta))$ that satisfies (1).*

Notice that this probability distribution, even with its sparsity guarantee, achieves the same dimension as Lemma 1. The proof of Theorem 2 presented in [18] involved complicated combinatorics; however, Cohen, Jayram, and Nelson [4] recently constructed two simple, combinatorics-free proofs of this result. The first approach, which is most relevant to the approach taken in this paper, used the Hanson-Wright bound on moments of quadratic forms. An analysis similar to the second approach can be recovered by specializing the analysis of Cohen [3] for sparse oblivious subspace embeddings to the case of “1-dimensional subspaces.” In fact, though this recovered analysis is more complex, it has the advantage of yielding dimension-sparsity tradeoffs that were not produced through any of the previous approaches: for $B \geq e$, the sparsity s can be set to $\Theta(\epsilon^{-1} \log_B(1/\delta))$ if m is set to $\Theta(B\epsilon^{-2} \log(1/\delta))$, enabling a $\log B$ factor reduction in sparsity at the expense of a B factor gain in dimension.

JL with sign-consistency constraints

Neuroscience-based constraints give rise to the additional condition of sign-consistency on the matrices in the probability distribution. Sign-consistency refers to the constraint that the nonzero entries of each column are either all positive or all negative. The relevance of dimensionality reduction schemes in neuroscience is described in a survey by Ganguli and Sompolinsky [9]. In convergent pathways in the brain, information stored in a massive number of neurons is compressed into a small number of neurons, and nonetheless the ability to perform the relevant computations is preserved. Modeling this information compression scheme requires a hypothesis regarding what properties of the original information must be accurately transmitted to the receiving neurons. A plausible minimum requirement is that convergent pathways preserve the similarity structure of neuronal representations at the source area.¹

It remains to select the appropriate mathematical measure of similarity. The candidate similarity measure considered in [9] is vector inner product, which conveniently gives rise to a model based on the JL distribution.² Suppose there are n “input” neurons at a source area and m “output” neurons at a target area. In this framework, the information at the input neurons is represented as a vector in \mathbb{R}^n , the synaptic connections to output neurons are represented as a $m \times n$ matrix (with (i, j) th entry corresponding to the strength of the connection between input neuron j and output neuron i), and the information received by the output neurons is represented as a vector in \mathbb{R}^m . The similarity measure between two

¹ This requirement is based on the experimental evidence that semantically similar objects in higher perceptual or association areas in the brain elicit similar neural activity patterns [19] and on the hypothesis that the similarity structure of the neural code is the basis of our ability to categorize objects and generalize responses to new objects [24].

² It is not difficult to see that for vectors x and y in the ℓ_2 unit ball, a $(1 + \epsilon)$ -approximation of $\|x\|_2$, $\|y\|_2$, and $\|x - y\|_2$ implies an additive error $\Theta(\epsilon)$ approximation of the inner product $\langle x, y \rangle$.

vectors v, w of neural information being taken to be $\langle v, w \rangle$ motivates modeling a synaptic connectivity matrix as a random $m \times n$ matrix drawn from a probability distribution that satisfies (1). Certain constraints on synaptic connectivity matrices arise from the biological limitations of neurons: the matrices must be *sparse* since a neuron is only connected to a small number (e.g. a few thousand) of postsynaptic neurons and *sign-consistent* since a neuron is usually purely excitatory or purely inhibitory.

This biological setting motivates the mathematical question: what is the optimal dimension and sparsity that can be achieved by a probability distribution over sparse, sign-consistent matrices that satisfies (1)? Allen-Zhu, Gelashvili, Micali, and Shavit [2] constructed a sparse, sign-consistent JL distribution³ and proved the following:

► **Theorem 3** (Sparse, sign-consistent JL [2]). *For every $\varepsilon > 0$, and $0 < \delta < 1/e$, there exists a probability distribution \mathcal{A} over $m \times n$ real, sign-consistent matrices with $m = \Theta(\varepsilon^{-2} \log^2(1/\delta))$ and sparsity $s = \Theta(\varepsilon^{-1} \log(1/\delta))$ that satisfies (1).*

In [2], it was also proven that the additional $\log(1/\delta)$ factor on m is essentially necessary: namely, any distribution over sign-consistent matrices satisfying (1) requires $m = \tilde{\Omega}(\varepsilon^{-2} \log(1/\delta) \min(\log(1/\delta), \log n))$. Thus, the dimension in Theorem 3 is essentially optimal. However, in order to achieve this upper bound on m , the proof presented in [2] involved complicated combinatorics even more delicate than in the analysis of sparse JL in [18].

We present a simpler, combinatorics-free proof of Theorem 3. Our analysis also yields dimension/sparsity tradeoffs, which were not previously known.⁴ We prove the following:

► **Theorem 4.** *For every $\varepsilon > 0$, $0 < \delta < 1$, and $e \leq B \leq \frac{1}{\delta}$, there exists a probability distribution \mathcal{A} over $m \times n$ real, sign-consistent matrices with $m = \Theta(B\varepsilon^{-2} \log_B^2(1/\delta))$ and sparsity $s = \Theta(\varepsilon^{-1} \log_B(1/\delta))$ that satisfies (1).*

Notice Theorem 3 is recovered if $B = e$. For larger B values, Theorem 4 enables a $\log B$ factor reduction in sparsity at the cost of a $B/\log^2 B$ factor gain in dimension.

To contextualize our tradeoff in Theorem 4, recall that the upper bounds on sparse (non-sign-consistent) JL dimension-sparsity tradeoffs by Cohen [3] take a similar form, allowing a $\log B$ factor reduction in s for a B factor gain in m . Moreover, in a recent follow-up work [13], we show lower bounds that indicate that the standard choice of sparse JL construction requires an exponential factor gain in dimension for a given reduction in sparsity, demonstrating that Cohen’s dimension-sparsity tradeoffs are essentially tight.⁵ Due to the structural similarity between sparse JL and sparse, sign-consistent JL, we believe this provides indication that our tradeoffs in Theorem 4 could be tight for this construction in many regimes.⁶

Proof Techniques

As in [2, 18, 4], our analysis is based on applying Markov’s inequality to the p th moment of an error term. Like in the first combinatorics-free analysis of sparse JL in [4], we express

³ Related mathematical work includes, in addition to sparse JL [18], a construction of a dense, sign-consistent JL distribution [23, 10].

⁴ In Appendix A, we point out the limiting lemma in the combinatorial analysis in [2], which prevents dimension-sparsity tradeoffs from being attainable through this approach, due to an assumption that is implicitly used in the analysis. For sparse JL, it is similarly not known how to obtain these tradeoffs via the combinatorial approach of [18].

⁵ More specifically, it follows from [3] and [13] that m is exactly $\min(\text{poly}(B)\varepsilon^{-2} \log(1/\delta), 2\varepsilon^{-2}/\delta)$ for the standard choice of sparse JL construction (uniformly choosing s nonzero entries per column).

⁶ An interesting direction for future work could be to build upon the ideas in the follow-up work [13] to show lower bounds on the dimension-sparsity tradeoffs for this sparse, sign-consistent JL construction.

this error term as a quadratic form of Rademachers (uniform ± 1 random variables), and our analysis then boils down to analyzing the moments of this quadratic form. While the analysis in [4] achieves the optimal dimension for sparse JL using an upper bound on the moments of quadratic forms of subgaussians due to Hanson and Wright [11], we give a counterexample in Section 3.2 that shows that the Hanson-Wright bound is too loose in the sign-consistent setting to result in the optimal dimension. Since the Hanson-Wright bound is tight for quadratic forms of gaussians, we thus require a separate treatment of quadratic forms of Rademachers.

We construct a simple bound on moments of quadratic forms of Rademachers that, unlike the Hanson-Wright bound, is sufficiently tight in our setting to prove Theorem 4. Our bound borrows some of the ideas from Latała’s tight bound on the moments of quadratic forms of Rademachers [21]. Although our bound is much weaker than the bound in [21] in the general case, it has the advantage of providing a greater degree of simplicity by consisting of easier-to-analyze terms; this simplicity is critical since our quadratic form coefficients are themselves random variables. The crux is that while the bound in [21] is focused on obtaining tight estimates for quadratic forms with scalar coefficients, our bound is much more tractable for quadratic forms with random variable coefficients. As a result, our bound enables a simple proof of Theorem 4, while retaining the necessary precision to recover the optimal dimension.

We build upon these ideas in our recent follow-up work [13], where the Hanson-Wright bound also turns out to be too loose. The work studies sparse JL performance in feature hashing and considers the restricted set of vectors with small ℓ_∞ -to- ℓ_2 norm ratio, continuing a line of work [27, 6, 17, 5, 18, 7]. The main result is a tight tradeoff between ℓ_∞ -to- ℓ_2 norm ratio and ϵ , δ , s , and m , and the lower bounds on dimension-sparsity tradeoffs mentioned before are shown as a corollary. Similar to this work, the proof boils down to a tight bound on p th moment of an error term, and it also turns out that the Hanson-Wright bound is too loose here. The work solves this issue by building upon ideas from this work, utilizing a separate treatment of Rademachers that is tractable for random variable coefficients. While the analysis in [13] does not use the exact quadratic form bound presented here, it uses intuition and generalizations of the moment bounding techniques presented in this work.

1.1 Notation

The main building blocks for our expressions are the following two types of random variables: Rademacher variables, which are uniform ± 1 random variables, and Bernoulli random variables, which have support $\{0, 1\}$. For any random variable X and value $p \geq 1$, we use the notation $\|X\|_p$ to denote the p -norm $(\mathbb{E}[|X|^p])^{1/p}$, where \mathbb{E} denotes the expectation. Similarly, for any random variable X and value $p \geq 1$ and any event E , we use the notation $\|X | E\|_p$ to denote the conditional p -norm $(\mathbb{E}[|X|^p | E])^{1/p}$, which is equivalent to the p -norm of the random variable $(X | E)$. We use the following notation to discuss certain asymptotics: given two scalar quantities Q_1 and Q_2 that are functions of some parameters, we use the notation $Q_1 \simeq Q_2$ to denote that there exist positive universal constants $C_1 \leq C_2$ such that $C_1 Q_2 \leq Q_1 \leq C_2 Q_2$, and we use the notation $Q_1 \lesssim Q_2$ to denote that there exists a positive universal constant C such that $Q_1 \leq C Q_2$.

1.2 A digression on Rademachers versus gaussians

The concept that drives our moment bound can be illustrated in the linear form setting. Suppose $\sigma_1, \sigma_2, \dots, \sigma_n$ are i.i.d Rademachers, $x = [x_1, \dots, x_n]$ is a vector in \mathbb{R}^n such that

$|x_1| \geq |x_2| \geq \dots \geq |x_n|$, and $2 \leq p \leq n$. The Khintchine inequality, which is tight for linear forms of gaussians, yields the ℓ_2 -norm bound $\|\sum_{i=1}^n \sigma_i x_i\|_p \lesssim \sqrt{p} \|x\|_2$. However, this bound *cannot* be a tight bound on $\|\sum_{i=1}^n \sigma_i x_i\|_p$ for the following reason: As $p \rightarrow \infty$, the quantity $\sqrt{p} \|x\|_2$ goes to infinity, while for any $p \geq 1$, the quantity $\|\sum_{i=1}^n \sigma_i x_i\|_p$ is bounded by $\|x\|_1$. Surprisingly, a result due to Hitczenko [12] indicates that the tight bound is actually the following combination of the ℓ_2 and ℓ_1 norm bounds:

$$\left\| \sum_{i=1}^n \sigma_i x_i \right\|_p \simeq \sum_{i=1}^p |x_i| + \sqrt{p} \sqrt{\sum_{i>p} x_i^2}.$$

In this bound, the “big” terms (i.e. terms involving x_1, x_2, \dots, x_p) are handled with an ℓ_1 -norm bound, while the remaining terms are approximated as gaussians and bounded with an ℓ_2 -norm bound.

A similar complication arises when the Hanson-Wright bound on quadratic forms of subgaussians is applied to Rademachers. Let σ be a d -dimensional vector of independent Rademachers, and let $A = (a_{k,l})$ be a symmetric $d \times d$ matrix with zero diagonal. The Hanson-Wright bound [11], which is tight for gaussians, states for any $p \geq 1$,

$$\|\sigma^T A \sigma\|_p \lesssim \sqrt{p} \sqrt{\sum_{k=1}^d \sum_{l=1}^d a_{k,l}^2} + p \left(\sup_{\|y\|_2=1} |y^T A y| \right).$$

Similar to the linear form setting, this bound *can't* be a tight bound on $\|\sigma^T A \sigma\|_p$ for the following reason: As $p \rightarrow \infty$, the quantity $\sqrt{p} \sqrt{\sum_{k=1}^d \sum_{l=1}^d a_{k,l}^2}$ goes to ∞ , while for any $p \geq 1$, the quantity $\|\sigma^T A \sigma\|_p$ is bounded by the entrywise ℓ_1 -norm $\sum_{k=1}^d \sum_{l=1}^d |a_{k,l}|$.

Our quadratic form bound is based on a degree-2 analog of Hitczenko’s observation. We analogously handle the “big” terms with an ℓ_1 -norm bound and bound the remaining terms by approximating some of the Rademachers by gaussians. From this, we obtain a combination of ℓ_2 and ℓ_1 norm bounds, similar to the linear form setting. Our simple bound has the surprising feature that it yields tighter guarantees than the Hanson-Wright bound yields for our error term. While our bound is weaker than Latała’s tight bound [21] on the moments of quadratic forms of Rademachers in the general case, it provides a greater degree of simplicity: our bound avoids an operator-norm-like term in Latała’s bound that is especially difficult to analyze when A is a random matrix, as is the case in this setting. Moreover, our bound still retains the necessary precision to recover the optimal dimension for sparse, sign-consistent JL.

Although our final analysis follows a style that this is perhaps less well-known within the TCS community, in the end, it is quite simple, relying only on our quadratic form bound coupled with a few standard tricks such as repeated use of triangle inequalities on $\|\cdot\|_p$ norms and standard moment bounds involving the binomial distribution. For this reason, we believe that it is likely to be of interest in other theoretical computer science settings involving moments or tail bounds of Rademacher forms.

1.3 Outline for the rest of the paper

In Section 2, we describe the construction and analysis of [2] for sparse, sign-consistent JL. In Section 3, we present the combinatorics-free approach in [4] for sparse JL that uses the Hanson-Wright bound, and we discuss why this approach does not yield the optimal dimension in the sign-consistent setting. In Section 4, we derive our bound on the moments

of quadratic forms of Rademachers and use this bound to construct a combinatorics-free proof of Theorem 4.

2 Existing Analysis for Sparse, Sign-Consistent JL

In Section 2.1, we describe how to construct the probability distribution of sparse, sign-consistent matrices analyzed in Theorem 3. In Section 2.2, we briefly describe the combinatorial proof of Theorem 3 presented in [2].

2.1 Construction of Sparse, Sign-Consistent JL

The entries of a matrix $A \in \mathcal{A}$ are generated as follows.⁷ Let $A_{i,j} = \eta_{i,j}\sigma_j/\sqrt{s}$ where $\{\sigma_i\}_{i \in [n]}$ and $\{\eta_{r,i}\}_{r \in [m], i \in [n]}$ are defined as follows:

- The families $\{\sigma_i\}_{i \in [n]}$ and $\{\eta_{r,i}\}_{r \in [m], i \in [n]}$ are independent from each other.
- The variables $\{\sigma_i\}_{i \in [n]}$ are i.i.d Rademachers.
- The variables $\{\eta_{r,i}\}_{r \in [m], i \in [n]}$ are identically distributed Bernoulli random variables with expectation s/m .
- The $\{\eta_{r,i}\}_{r \in [m], i \in [n]}$ are independent across columns but not independent within each column. For every column $1 \leq i \leq n$, it holds that $\sum_{r=1}^m \eta_{r,i} = s$. For every subset $S \subseteq [m]$ and every column $1 \leq i \leq n$, it holds that $\mathbb{E}[\prod_{r \in S} \eta_{r,i}] \leq \prod_{r \in S} \mathbb{E}[\eta_{r,i}]$. (One common definition of $\{\eta_{r,i}\}_{r \in [m], i \in [n]}$ that satisfies these conditions is the distribution defined by uniformly choosing exactly s of these variables per column to be a 1.)

For every $x \in \mathbb{R}^n$ such that $\|x\|_2 = 1$, we need to analyze an error term, which for this construction is the following random variable:

$$Z := \|Ax\|_2^2 - 1 = \frac{1}{s} \sum_{i \neq j} \sum_{r=1}^m \eta_{r,i} \eta_{r,j} \sigma_i \sigma_j x_i x_j.$$

Proving that \mathcal{A} satisfies (1) boils down to proving that $\mathbb{P}_{\eta, \sigma}[|Z| > \epsilon] < \delta$. The main technique to prove this tail bound is the moment method. Bounding a large moment of Z is useful since it follows from Markov's inequality that

$$\mathbb{P}_{\eta, \sigma}[|Z| > \epsilon] = \mathbb{P}_{\eta, \sigma}[|Z|^p > \epsilon^p] < \frac{\mathbb{E}[|Z|^p]}{\epsilon^p}.$$

The usual approach, used in the analyses in [2, 18, 4] as well as in our analysis, is to take $p = \Theta(\log(1/\delta))$ to be an even integer and analyze the p -norm $\|Z\|_p$ of the error term.

2.2 Discussion of the combinatorial analysis of [2]

In the analysis in [2], a complicated combinatorial argument was used to prove the following lemma, from which Theorem 3 follows:

► **Lemma 5** ([2]). *If $s^2 \leq m$ and $p < s$, then $\|Z\|_p \lesssim \frac{p}{s}$.*

The argument in [2] to prove Lemma 5 was based on expanding $\mathbb{E}[Z^p]$ into a polynomial with $\approx n^{2p}$ terms, establishing a correspondence between the monomials and the multigraphs, and then doing combinatorics to analyze the resulting sum. The approach of mapping monomials to graphs is commonly used in analyzing the eigenvalue spectrum of random matrices [28, 8]

⁷ See Appendix B for a formal construction of the probability space.

and was also used in [18] to analyze sparse JL. The analysis in [2] borrowed some methods from the analysis in [18]; however, the additional correlations between the Rademachers imposed by sign-consistency forced the analysis in [2] to require more delicate manipulations at several stages of the computation.

The expression to be analyzed was $s^p \mathbb{E}[Z^p]$, which was written as:

$$\sum_{i_1, \dots, i_p, j_1, \dots, j_p \in [n], i_1 \neq j_1, \dots, i_p \neq j_p} \left(\prod_{u=1}^p x_{i_u} x_{j_u} \right) \left(\mathbb{E}_\sigma \prod_{u=1}^p \sigma_{i_u} \sigma_{j_u} \right) \left(\mathbb{E}_\eta \prod_{u=1}^t \sum_{r=1}^m \eta_{r, i_u} \eta_{r, j_u} \right).$$

After layers of computation, it was shown that

$$s^p \mathbb{E}[Z^p] \leq e^p \sum_{v=2}^p \sum_{G \in \mathcal{G}_{v,p}} \left((1/p^p) \prod_{q=1}^v \sqrt{d_q}^{d_q} \right) \sum_{r_1, \dots, r_p \in [m]} \prod_{i=1}^w (s/m)^{v_i}$$

where $\mathcal{G}_{v,p}$ is a set of directed multigraphs with v labeled vertices and t labeled edges, where d_q is the total degree of vertex $q \in [v]$ in a graph $\mathcal{G}_{v,p}$, and where w and v_1, \dots, v_w are defined by G and the edge colorings r_1, \dots, r_t . The problem then boiled down to carefully enumerating the graphs in $\mathcal{G}_{v,p}$ in six stages and analyzing the resulting expression.

3 Discussion of Combinatorics-Free Approaches

The main ingredient of the first combinatorics-free approach for sparse JL presented in [4] is the Hanson-Wright bound on the moments of quadratic forms of subgaussians. In Section 3.1, we discuss the approach in [4]. In Section 3.2, we discuss why this approach, if applied to sparse, sign-consistent JL, fails to yield the optimal dimension.

3.1 Hanson-Wright approach for sparse JL in [4]

The relevant random variable for sparse JL is

$$Z' = \|Ax\|^2 - 1 = \frac{1}{s} \sum_{r=1}^m \sum_{i \neq j} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} x_i x_j$$

where the n independent Rademachers $\{\sigma_i\}_{i \in [n]}$ from the sign-consistent case are replaced by the mn independent Rademachers $\{\sigma_{r,i}\}_{i \in [n], r \in [m]}$. The main idea in [4] was to view Z' as a quadratic form $\frac{1}{s} \sigma^T A \sigma$. Here, σ is a mn -dimensional vector of independent Rademachers and $A = (A_{k,l})$ is a symmetric, zero diagonal, block diagonal $mn \times mn$ matrix with m blocks of size $n \times n$, where the (i, j) th entry (for $i \neq j$) of the r th block is $\eta_{r,i} \eta_{r,j} x_i x_j$. The quantity $\|\sigma^T A \sigma\|_p$ was analyzed using the Hanson-Wright bound. In order to bound $\|\sigma^T A \sigma\|_p$, since A is a random matrix whose entries depend on the η values, an expectation had to be taken over η in the expression given by the Hanson-Wright bound. This resulted in the following:

$$\|\sigma^T A \sigma\|_p \lesssim \left\| \sqrt{p} \sqrt{\sum_{k=1}^{mn} \sum_{l=1}^{mn} A_{k,l}^2} + p \sup_{\|y\|_2=1} |y^T A y| \right\|_p. \quad (2)$$

The remainder of the analysis boiled down to bounding the RHS of (2), and it successfully recovered Theorem 2.

3.2 Failure of the Hanson-Wright approach for sparse, sign-consistent JL

The Hanson-Wright-based approach for sparse JL in [4] cannot be applied to the sign-consistent case to obtain a tight bound on $\|Z\|_p$. The loss arises from the fact that while the Hanson-Wright bound is tight for quadratic forms of Gaussians, it is not guaranteed to be tight for quadratic forms of Rademachers. As discussed in Section 1.2, when $p \rightarrow \infty$, the Hanson-Wright bound goes to ∞ , while $\|\sigma^T A \sigma\|_p$ can be bounded by the entrywise ℓ_1 norm of the matrix A . Although approximating the error term Rademachers by Gaussians happened to be sufficiently tight for sparse JL, this loss results in a suboptimal dimension for sparse, sign-consistent JL.⁸ We give a counterexample, i.e. a vector x , that shows that the Hanson-Wright bound is too loose to give the optimal dimension (when $\{\eta_{r,i}\}_{r \in [m], i \in [n]}$ are defined by uniformly choosing exactly s of the variables per column to be a 1). We present the details in Appendix F.

4 Simple Proof of Theorem 4

The main ingredient in our proof of Theorem 4 is the following bound on $\|Z\|_p$:

► **Lemma 6.** *Let $B = m/s^2$. If $p \geq 2$, then*

$$\|Z\|_p \lesssim \begin{cases} \frac{p}{s \log B}, & \text{if } B \geq e \\ \frac{p}{sB} & \text{if } B < e. \end{cases}$$

We will later show that Theorem 4 follows from Lemma 6 via Markov's inequality.

In order to analyze $\|Z\|_p$, we view Z as a quadratic form $\frac{1}{s} \sigma^T A \sigma$, where the vector σ is an n -dimensional vector of independent Rademachers, and $A = (a_{i,j})$ is a symmetric, zero-diagonal $n \times n$ matrix where the (i, j) th entry (for $i \neq j$) is $x_i x_j \sum_{r=1}^m \eta_{r,i} \eta_{r,j}$. Since Z is symmetric in x_1, \dots, x_n , we can assume WLOG that $|x_1| \geq |x_2| \geq \dots \geq |x_n|$. For convenience, we define, like in [4],

$$Q_{i,j} := \sum_{r=1}^m \eta_{r,i} \eta_{r,j} \tag{3}$$

to be the number of collisions between the nonzero entries of the i th column and the nonzero entries of the j th column. Now, the (i, j) th entry of A (for $i \neq j$) can be written as $Q_{i,j} x_i x_j$.

As discussed in Section 3.2, we cannot apply the Hanson-Wright bound to tightly analyze $\|Z\|_p$ and thus require a separate treatment of Rademachers. We derive the following moment bound on quadratic forms of Rademachers⁹ that yields tighter guarantees than the Hanson-Wright bound yields for $\|Z\|_p$:

⁸ The difference results from the correlations between the signs resulting in more ‘‘tightly packed’’ coefficients in the error term quadratic form in the sign-consistent case.

⁹ As mentioned before, Latała [21] provides a tight bound on the moments of $\sigma^T A \sigma$ (and on the moments of more general quadratic forms). However, his bound consists of terms that are difficult to analyze when the quadratic form coefficients are random variables. Moreover, his proof is quite complicated, though the bound can be used in a black box to generate a much messier solution (by unravelling some of his proof to avoid the operator-norm-like term).

► **Lemma 7.** *If $A = (a_{i,j})$ is a symmetric square $n \times n$ matrix with zero diagonal, $\{\sigma_i\}_{i \in [n]}$ is a set of independent Rademachers, and $q \geq 1$, then*

$$\left\| \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \sigma_i \sigma_j \right\|_q \lesssim \left(\sum_{i=1}^{\min(q,n)} \sum_{j=1}^{\min(q,n)} |a_{i,j}| \right) + \sqrt{q} \sqrt{\sum_{i=1}^n \left\| \sum_{j>q} a_{i,j} \sigma_j \right\|_q^2}.$$

Observe that our bound avoids the weakness of the Hanson-Wright bound in the limit as $q \rightarrow \infty$. As discussed in Section 1.2, $\left\| \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \sigma_i \sigma_j \right\|_q$ can be bounded by the entrywise ℓ_1 -norm bound $\sum_{i=1}^n \sum_{j=1}^n |a_{i,j}|$ for any $q \geq 1$. While the Hanson-Wright bound goes to ∞ as $q \rightarrow \infty$, the bound in Lemma 7 approaches the entrywise ℓ_1 bound in the limit: for $q > n$, the second term in Lemma 7 vanishes since the summand $\sum_{j>q}$ is empty. As a result, the bound becomes the first-term, which becomes $\sum_{i=1}^n \sum_{j=1}^n |a_{i,j}|$ as desired. For $1 \leq q < n$, our bound becomes an interpolation of ℓ_1 and ℓ_2 norm bounds that bears resemblance to Hitzchenko's Rademacher linear form bound in [12] discussed in Section 1.2.

Although our bound is weaker than Latała's bound in [21] in the general case, it is much simpler to analyze, especially when A is a random matrix. While the bound in [21] is focused on obtaining tight estimates for quadratic forms where A is a scalar matrix, our bound is much more tractable when A is a random matrix. The main complication in the bound in [21] arises from the operator-norm-like term $\sup_{\|y\|_2=1, \|y\|_\infty \leq \frac{1}{\sqrt{q}}} |y^T A y|$. Due to the asymmetrical geometry of the ℓ_2 ball truncated by ℓ_∞ planes, this term becomes especially messy in our setting when A is a random matrix. Observe that our bound in Lemma 7 manages to avoid this term altogether. Moreover, our ℓ_1 norm term is straightforward to calculate, and our ℓ_2 norm term can be handled cleanly through a bound (Lemma 15) from [20] on the q -norm $\left\| \sum_{j>q} a_{i,j} \sigma_j \right\|_q$ that is tractable even when the $a_{i,j}$ are themselves random variables.

We defer our proof of Lemma 7 to Section 4.1. We now use Lemma 7 and the triangle inequality to obtain the following bound on $\|Z\|_p$:

$$\begin{aligned} \|Z\|_p &= \frac{1}{s} \left(\mathbb{E}_\eta \mathbb{E}_\sigma \left[\sum_{i=1}^n \sum_{j \leq n, j \neq i} Q_{i,j} x_i x_j \sigma_i \sigma_j \right]^p \right)^{1/p} \\ &\lesssim \frac{1}{s} \left(\mathbb{E}_\eta \left[\sum_{i=1}^p \sum_{\substack{j \leq p \\ j \neq i}} |Q_{i,j} x_i x_j| + \sqrt{p} \sqrt{\sum_{i=1}^n \left(\mathbb{E}_\sigma \left[\sum_{\substack{j>p \\ j \neq i}} Q_{i,j} x_i x_j \sigma_j \right]^p \right)^{2/p}} \right]^p \right)^{1/p} \\ &\leq \frac{1}{s} \left(\underbrace{\left\| \sum_{i=1}^p \sum_{\substack{j \leq p \\ j \neq i}} |Q_{i,j} x_i x_j| \right\|_p}_{(*)} + \sqrt{p} \underbrace{\sqrt{\sum_{i=1}^n \left\| \sum_{\substack{j>p \\ j \neq i}} Q_{i,j} x_i x_j \sigma_j \right\|_p^2}}_{(**)} \right). \end{aligned}$$

We first discuss some intuition for why using this bound to analyze $\|Z\|_p$ avoids the loss incurred by the Hanson-Wright bound here. In the Hanson-Wright bound, all of the Rademachers are essentially approximated by Gaussians. In our bound, we make use of Rademachers in the appropriate places to avoid loss. For $1 \leq i \leq p$ and $1 \leq j \leq p$ (the upper left $p \times p$ minor where the $|x_i|$ and $|x_j|$ values are the largest), our approach utilizes

an ℓ_1 -norm bound rather than \sqrt{p} times an ℓ_2 bound, which turns out to allow us to save a factor of \sqrt{p} in the resulting bound on $\|Z\|_p$. Now, since the original matrix is symmetric, it only remains to consider $1 \leq i \leq n$ and $p+1 \leq j \leq n$. In this range, we approximate the σ_i Rademachers by gaussians and use an ℓ_2 -norm bound. It turns out that approximating the σ_j Rademachers by gaussians as well would yield too loose of a bound for our application, so we preserve the σ_j Rademachers. For the remaining Rademacher linear forms, the interaction between the x_j values (all of which are upper bounded in magnitude by $\frac{1}{\sqrt{p}}$) and the σ_j Rademachers yields the desired bound.

In order to prove Lemma 6, it remains to prove Lemma 7 as well as to bound (*) and (**). In Section 4.1, we prove Lemma 7. In Section 4.2 and Section 4.3, we bound (*) and (**). Since the building blocks of (*) and (**) are weighted sums of the $Q_{i,j}$ random variables, we first bound moments of these random variables separately. In Section 4.2, we use the binomial-like properties of the $Q_{i,j}$ s coupled with standard moment bounds involving the binomial distribution to analyze the moments. In Section 4.3, we use these moment bounds to bound (*) and (**), and then finish our proof of Lemma 6. In Section 4.4, we show how Lemma 6 implies Theorem 4.

4.1 Proof of Lemma 7

We use the following standard lemmas in our proof of Lemma 7.

The first lemma allows us to decouple the two sets of Rademachers in our quadratic form so that we can reduce analyzing the moments of the quadratic form to analyzing the moments of a linear form.

► **Lemma 8** (Decoupling, Theorem 6.1.1 of [26]). *If $A = (a_{i,j})$ is a symmetric, zero-diagonal $n \times n$ matrix and $\{\sigma_i\}_{i \in [n]} \cup \{\sigma'_i\}_{i \in [n]}$ are independent Rademachers, then*

$$\left\| \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \sigma_i \sigma_j \right\|_q \lesssim \left\| \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \sigma'_i \sigma_j \right\|_q.$$

The next lemma is due to Khintchine and gives an ℓ_2 -norm bound on linear forms of Rademachers. Since the Khintchine bound is derived from approximating $\sigma_1, \dots, \sigma_n$ by i.i.d gaussians, we only use this bound outside of the upper left $p \times p$ minor of our matrix A .

► **Lemma 9** (Khintchine). *If $\sigma_1, \sigma_2, \dots, \sigma_n$ are independent Rademachers, then for all $q \geq 1$ and $a \in \mathbb{R}^n$,*

$$\left\| \sum_{i=1}^n \sigma_i a_i \right\|_q \lesssim \sqrt{q} \|a\|_2.$$

Now, we are ready to prove Lemma 7.

Proof of Lemma 7. By Lemma 8 and the triangle inequality, we know

$$\left\| \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \sigma_i \sigma_j \right\|_q \lesssim \underbrace{\left\| \sum_{i=1}^{\min(q,n)} \sum_{j=1}^{\min(q,n)} a_{i,j} \sigma'_i \sigma_j \right\|_q}_{\alpha} + \underbrace{\left\| \sum_{i=1}^n \sum_{j>q} a_{i,j} \sigma'_i \sigma_j \right\|_q}_{\beta} + \underbrace{\left\| \sum_{i>q} \sum_{j=1}^q a_{i,j} \sigma'_i \sigma_j \right\|_q}_{\gamma}.$$

We first bound α . Since a Rademacher σ satisfies $|\sigma| = 1$, it follows that α can be upper bounded by the entrywise ℓ_1 -norm bound $\sum_{i=1}^{\min(q,n)} \sum_{j=1}^{\min(q,n)} |a_{i,j}|$ as desired. Using Lemma 9, we know that β can be upper bounded by:

$$\sqrt{q} \left\| \sqrt{\sum_{i=1}^n \left(\sum_{j>q} a_{i,j} \sigma_j \right)^2} \right\|_q = \sqrt{q} \sqrt{\left\| \sum_{i=1}^n \left(\sum_{j>q} a_{i,j} \sigma_j \right)^2 \right\|_{q/2}} \leq \sqrt{q} \sqrt{\sum_{i=1}^n \left\| \sum_{j>q} a_{i,j} \sigma_j \right\|_q^2}.$$

We now bound γ . An analogous argument shows $\gamma \leq \sqrt{q} \sqrt{\sum_{j=1}^q \left\| \sum_{i>q} a_{i,j} \sigma_i \right\|_q^2}$. Thus:

$$\gamma \leq \sqrt{q} \sqrt{\sum_{j=1}^q \left\| \sum_{i>q} a_{i,j} \sigma_i \right\|_q^2} \leq \sqrt{q} \sqrt{\sum_{j=1}^n \left\| \sum_{i>q} a_{i,j} \sigma_i \right\|_q^2} = \sqrt{q} \sqrt{\sum_{i=1}^n \left\| \sum_{j>q} a_{i,j} \sigma_j \right\|_q^2}.$$

◀

4.2 Moments of Weighted Sums of $Q_{i,j}$ Random Variables

Recall that for $1 \leq i \neq j \leq n$, the $Q_{i,j}$ random variables count the number of collisions between the nonzero entries in the i th column and j th column. We first prove that these sets of random variables satisfy (conditional) independence properties, when conditioned on any choice of nonzero entries in the i th column. We also show that the moments of the random variables obtained through this conditioning are bounded by binomial moments.

► **Proposition 10.** *Let X be a random variable distributed as $\text{Bin}(s, s/m)$. For any $1 \leq i \leq n$, given any choice of s nonzero rows $r_1 \neq r_2 \neq \dots \neq r_s$ in the i th column, the set of $n-1$ random variables¹⁰ $\{(Q_{i,j} \mid \eta_{r_1,i} = \eta_{r_2,i} = \dots = \eta_{r_s,i} = 1)\}_{1 \leq j \leq n, j \neq i}$ are independent. Moreover, for any $q \geq 1$ and any $j \neq i$:*

$$\|Q_{i,j} \mid \eta_{r_1,i} = \eta_{r_2,i} = \dots = \eta_{r_s,i} = 1\|_q \leq \|X\|_q.$$

The independence properties use that the nonzero entries in different columns are independent. Moreover, the binomial bound on the moments of $Q_{i,j}$ follows from the decomposition of $Q_{i,j} \mid \eta_{r_1,i} = \eta_{r_2,i} = \dots = \eta_{r_s,i} = 1$ into a sum of Bernoulli random variables.

Proof of Proposition 10. Let A be a matrix drawn from \mathcal{A} , and pick any $1 \leq i \leq n$. We condition on the event that the s nonzero entries in column i of A occur at rows r_1, \dots, r_s . For $1 \leq j \leq n$, $j \neq i$ and $1 \leq k \leq s$, let $Y_{k,j} = \eta_{r_k,j}$, so that $(Q_{i,j} \mid \eta_{r_1,i} = \eta_{r_2,i} = \dots = \eta_{r_s,i}) = \sum_{k=1}^s Y_{k,j}$. Notice that the sets $\{Y_{k,j}\}_{k \in [s]}$ for $1 \leq j \leq n$, $j \neq i$ are independent from each other, which means random variables in the set $\{Q_{i,j} \mid \eta_{r_1,i} = \eta_{r_2,i} = \dots = \eta_{r_s,i} = 1\}_{1 \leq j \leq n, j \neq i}$ are independent. For $1 \leq j \leq n$, $j \neq i$, and $1 \leq k \leq s$, let $Z_{k,j}$ be distributed as i.i.d Bernoulli random variables with expectation s/m . Notice that for a fixed j , each $Y_{k,j}$ is distributed as $Z_{k,j}$ and the random variables $\{Y_{k,j}\}_{1 \leq k \leq s}$ are negatively correlated (and nonnegative), which means

$$\|Q_{i,j} \mid \eta_{r_1,i} = \eta_{r_2,i} = \dots = \eta_{r_s,i} = 1\|_q = \left\| \sum_{k=1}^s Y_{k,j} \right\|_q \leq \left\| \sum_{k=1}^s Z_{k,j} \right\|_q = \|X\|_q.$$

◀

¹⁰ See Appendix B for a formal discussion of viewing these quantities as random variables over a different probability space.

Now, we need to analyze the moments of weighted sums of $Q_{i,j}$ random variables. Using the independence properties and the fact that the moments of the $Q_{i,j}$ are upper bounded by binomial moments as given in Proposition 10, this boils down to studying the moments of weighted sums of binomial random variables. The main tools that we use in analyzing these moments are bounds on moments of sums of nonnegative random variables and sums of symmetric random variables due to Latała [20] that we state in Appendix C.¹¹

Our first estimate is an upper bound on the moments of binomial random variables, which also gives bounds on moments of the $Q_{i,j}$ by Proposition 10. We defer the proof to Appendix E.

► **Proposition 11.** *Suppose that X is a random variable distributed as $\text{Bin}(N, \alpha)$ for any $\alpha \in (0, 1)$ and any integer $N \geq 1$. If $q \geq 1$ and $B = \frac{q}{\alpha \max(N, q)}$, then*

$$\|X\|_q \lesssim \begin{cases} \frac{q}{\log B} & \text{if } B \geq e \\ \frac{q}{B} & \text{if } B < e \end{cases}.$$

Our next estimate is essentially an upper bound on the moments of sums of binomial random variables weighted by Rademachers. We defer the proof to Appendix D.

► **Proposition 12.** *Suppose that $q \geq 2$ is even and $y = [y_1, \dots, y_M]$ is a vector that satisfies $\|y\|_2 \leq 1$ and $\|y\|_\infty \leq \frac{1}{\sqrt{q}}$. Let X be a random variable distributed as $\text{Bin}(N, \alpha)$ for some $\alpha \in (0, 1)$ and some integer $N \geq 1$. Suppose that Y_1, \dots, Y_M are independent random variables that satisfy $\|Y_k\|_l \leq \|X\|_l$ for $1 \leq k \leq M$ and for $l \geq 1$. Suppose that $\sigma_1, \dots, \sigma_M$ are independent Rademachers, also independent of $\{Y_k\}_{k \in [M]}$. If $B = \frac{1}{\alpha N}$, then*

$$\left\| \sum_{k=1}^M Y_k y_k \sigma_k \right\|_q \lesssim \begin{cases} \frac{\sqrt{q}}{\log B} & \text{if } B \geq e \\ \frac{\sqrt{q}}{B} & \text{if } B < e \end{cases}.$$

4.3 Bounding (*) and (**) to prove Lemma 6

We bound the quantities (*) and (**) in the following sublemmas, which assume the notation used throughout the paper:

► **Lemma 13.** *If $m/s^2 = B$, then*

$$\left\| \sum_{i=1}^p \sum_{j \leq p, j \neq i} |Q_{i,j} x_j x_i| \right\|_p \lesssim \begin{cases} \frac{p}{\log B} & \text{if } B \geq e \\ \frac{p}{B} & \text{if } B < e \end{cases}.$$

► **Lemma 14.** *If $m/s^2 = B$, then*

$$\sqrt{p} \sqrt{\sum_{i=1}^n \left\| \sum_{j > p, j \neq i} Q_{i,j} x_i x_j \sigma_j \right\|_p^2} \lesssim \begin{cases} \frac{p}{\log B} & \text{if } B \geq e \\ \frac{p}{B} & \text{if } B < e \end{cases}.$$

We now use Proposition 10 as well as the moment bound on binomial random variables from Proposition 11 to prove Lemma 13 and thus bound (*).

¹¹The proofs of these bounds given in [20] are not complicated; for the sake of being self-contained, we give sketches of these proofs in Appendix C.

Proof of Lemma 13. We carefully use the triangle inequality to see¹²:

$$\left\| \sum_{i=1}^p \sum_{\substack{j \leq p \\ j \neq i}} |Q_{i,j} x_j x_i| \right\|_p \leq 2 \left\| \sum_{i=1}^p \sum_{\substack{j \leq p \\ j > i}} Q_{i,j} |x_j| |x_i| \right\|_p \lesssim \left\| \sum_{i=1}^p x_i^2 \sum_{\substack{j \leq p \\ j > i}} Q_{i,j} \right\|_p \lesssim \sum_{i=1}^p x_i^2 \left\| \sum_{\substack{j \leq p \\ j > i}} Q_{i,j} \right\|_p.$$

Let $X \sim \text{Bin}(s, s/m)$ and $Y \sim \text{Bin}(sp, s/m)$. By Proposition 10, for any i and any $r_1 \neq r_2 \neq \dots \neq r_s$, the random variables $\{Q_{i,j} \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1\}_{j \neq i}$ are independent and $\|Q_{i,j} \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1\|_p \leq \|X\|_p$. It follows from taking p th powers of both sides that

$$\left\| \left(\sum_{\substack{j \leq p \\ j > i}} Q_{i,j} \right) \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1 \right\|_p = \left\| \sum_{\substack{j \leq p \\ j > i}} (Q_{i,j} \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1) \right\|_p \leq \|Y\|_p.$$

Now, Proposition 11 gives us a bound on $\|Y\|_p$, and the result follows from the law of total expectation.¹³ \blacktriangleleft

We now use Proposition 10 as well as the moment bound on weighted sums of binomial random variables from Proposition 12 to prove Lemma 14 and thus bound (**).

Proof of Lemma 14. Observe that

$$\sqrt{p} \sqrt{\sum_{i=1}^n \left\| \sum_{\substack{j > p \\ j \neq i}} Q_{i,j} x_i x_j \sigma_j \right\|_p^2} = \sqrt{p} \sqrt{\sum_{i=1}^n x_i^2 \left\| \sum_{\substack{j > p \\ j \neq i}} Q_{i,j} x_j \sigma_j \right\|_p^2} \leq \sqrt{p} \max_{1 \leq i \leq n} \left\| \sum_{\substack{j > p \\ j \neq i}} Q_{i,j} x_j \sigma_j \right\|_p.$$

Let $X \sim \text{Bin}(s, s/m)$. By Proposition 10, for any i and any $r_1 \neq r_2 \neq \dots \neq r_s$, the random variables $\{Q_{i,j} \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1\}_{j \neq i}$ are independent and $\|Q_{i,j} \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1\|_p \leq \|X\|_p$. Moreover, $|x_j| \leq \frac{1}{\sqrt{p}}$ for $j > p$. Now, we consider $\left\| \sum_{j > p, j \neq i} Q_{i,j} x_j \sigma_j \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1 \right\|_p$ which is equal to

$$\left\| \sum_{j > p, j \neq i} (Q_{i,j} \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1) (\sigma_j \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1) x_j \right\|_p.$$

Since each $(\sigma_j \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1)$ is distributed as a Rademacher and since the set of $n-1$ random variables $\{\sigma_j \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1\}_{j \neq i}$ are independent and also independent of $\{Q_{i,j} \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1\}_{j \neq i}$, we can apply Proposition 12 to this expression and thus get a bound¹⁴ on the conditional p -norm $\left\| \sum_{j > p, j \neq i} Q_{i,j} x_j \sigma_j \mid \eta_{r_1,i} = \dots = \eta_{r_s,i} = 1 \right\|_p$. Now, the result follows from the law of total expectation. \blacktriangleleft

We now show the bound on $\|Z\|_p$ follows from the bounds on (*) and (**) in Lemmas 13, 14.

¹²Naively applying the triangle inequality yields a suboptimal bound, so we require this more careful treatment.

¹³See Appendix B for a formal discussion of why a uniform bound on the conditional p -norm implies a bound on the p -norm here.

¹⁴Approximating the σ_j by gaussians yields a suboptimal bound, so we require the bound given in Proposition 12.

Proof of Lemma 6. Applying Lemmas 13, 14 after the following simplification proves the lemma:

$$\|Z\|_p \lesssim \frac{1}{s} \left\| \sum_{i=1}^p \sum_{j \leq p, j \neq i} |Q_{i,j} x_i x_j| \right\|_p + \frac{\sqrt{p}}{s} \sqrt{\sum_{i=1}^n \left\| \sum_{j > p, j \neq i} Q_{i,j} x_i x_j \sigma_j \right\|_p^2}.$$

◀

4.4 Proof of Theorem 4

We show Lemma 6 implies Theorem 4, completing the proof.

Proof of Theorem 4. It suffices to show $\mathbb{P}_{\eta, \sigma}[|Z| > \epsilon] < \delta$. By Markov's inequality, we know

$$\mathbb{P}_{\eta, \sigma}[|Z| > \epsilon] = \mathbb{P}_{\eta, \sigma}[|Z|^p > \epsilon^p] < \epsilon^{-p} \mathbb{E}[|Z|^p] = \left(\frac{\|Z\|_p}{\epsilon} \right)^p.$$

Suppose that $B \geq e$. Then by Lemma 6, we know

$$\left(\frac{\|Z\|_p}{\epsilon} \right)^p \leq \left(\frac{Cp}{(\log B)s\epsilon} \right)^p.$$

Thus, to upper bound this quantity by δ , we can set $s = \Theta(\epsilon^{-1}p/\log B) = \Theta(\epsilon^{-1} \log_B(1/\delta))$ and $m = \Theta(Bs^2)$. We impose the additional constraint that $B \leq \frac{1}{\delta}$ to guarantee that $s \geq 1$. This proves the desired result.¹⁵

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¹⁵If we set $B < e$, if we use Lemma 6, we know that in order to obtain an upper bound of δ , we would have to set $s = \Theta(\epsilon^{-1}p/B) = \Theta(\epsilon^{-1} \log(1/\delta)/B)$ and $m = \Theta(\epsilon^{-1} \log^2(1/\delta)/B)$. This yields no better s or m values than those achieved when $B = e$.

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A Limitations of the Combinatorial Approach

At first glance, it appears that the bound of $\mathbb{P}[|Z| > \epsilon] \leq \left(\frac{Cts}{\epsilon m}\right)^t$ of [2] in the proof of the main theorem (p. 9 of the arXiv version) implies the desired dimension-sparsity tradeoffs by setting $s = \epsilon^{-1}t$, $m = \frac{B\epsilon^{-1}ts}{2C}$, and $t = \log_B(1/\delta)$ (this t value is equivalent to the p value in this paper). However, this does not actually follow from the analysis in [2]: there is an assumption made in one of the lemmas, which is not stated explicitly in the statement of the lemma, that does not allow the parameters to be set in this way. The limiting factor is the lemma that states that

$$s^t \mathbb{E}[Z^t] \leq 2^{O(t)} t^t \left(\frac{s^2}{m}\right)^t.$$

This is Lemma 3 in the conference version of [2], and Lemma 4.3 in the arXiv version of [2]. Here, Z is defined analogously as in section 2.1 of this paper.

The proof of this lemma, given in Appendix A.3 in [2], implicitly relies on the fact that $\frac{s^2}{m} \geq 1$, although this condition is not explicitly stated in the lemma statement. This assumption arises from the last line of the proof, where the sum $\sum_{w=1}^t \left(\frac{s^2}{m}\right)^w$ is upper bounded by $t \left(\frac{s^2}{m}\right)^t$. Following the end of the proof of Theorem 1 (the top of p. 9 of the arXiv version), this yields $\mathbb{P}[|Z| > \epsilon] \leq \left(\frac{Cts}{\epsilon m}\right)^t$. Now, suppose we instead set $m = Bs^2$ (where $B \leq 1$ as required by the assumption). Then we obtain $\left(\frac{Cts}{\epsilon m}\right)^t = \left(\frac{Ct}{\epsilon Bs}\right)^t$. Thus, we can set s to be $C\epsilon^{-1}B^{-1} \log(1/\delta)$ and m to be $C^2\epsilon^{-2} \log^2(1/\delta)B^{-1}$. Since $B \leq 1$, this is no better than the original theorem statement and thus yields no dimension-sparsity tradeoff.

Now, suppose we instead let $\frac{s^2}{m} \leq 1$. Then we can modify the proof of Lemma 4.3 to obtain the weaker upper bound of $\sum_{w=1}^t \left(\frac{s^2}{m}\right)^w$ by $t \frac{s^2}{m}$. Let $B = m/s^2$ where $B \geq 1$. In order to ensure that m is polynomial in $\log(1/\delta)$, we assume that $B \leq \delta$. In this case, mimicking the calculation at the end of the proof of Theorem 1, we obtain $\mathbb{P}[|Z| > \epsilon] \leq \frac{1}{B} \left(\frac{Ct}{\epsilon s}\right)^t = \left(\frac{Ct}{\epsilon s B^{1/t}}\right)^t$. Thus, we can set $s = C \log(1/\delta) \epsilon^{-1} e^{-\log B/t}$. Observe that $0 \leq \log B \leq t$, so $1 \geq e^{-\log B/t} \geq e^{-1}$. Thus, $s = \Theta(\log(1/\delta) \epsilon^{-1})$ and $m = \Theta(Bs^2)$, which does not yield a dimension-sparsity tradeoff.

Thus, it is not clear how to directly obtain the dimension-sparsity tradeoff from the combinatorial approach of [2]. Some intuition for this limitation is that the moment bounds on Z obtained by the combinatorial approach are not sufficiently tight for varying values of B due to the fact that the bounding techniques are implicitly tailored to the case of $B = \Theta(1)$. The combinatorics-free approach in this paper avoids this issue through making use of a more structured method to bound the moments of Z .

B Formal Definition of Probability Space and Random Variables

B.1 Construction of the Probability Space

Rather than defining our probability space over $m \times n$ sparse, sign-consistent matrices implicitly by random variables, we explicitly define this (finite) probability space here. (For all probability spaces defined, the set of events will be all subsets of the sample space.) We take our probability space to be the product space of n probability spaces $(\Omega_i, \Sigma_i, \mathbb{P}_i)$ where Ω_i is the set of m -dimensional column vectors with entries in $\{-1/\sqrt{s}, 0, 1/\sqrt{s}\}$ with exactly s nonzero entries and all nonzero entries the same sign. Thus, the product $\Omega := \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ is in one-to-one correspondence with the sample space of $m \times n$

dimensional sign-consistent matrices with entries in $\{-1/\sqrt{s}, 0, 1/\sqrt{s}\}$ with exactly s nonzero entries in each column. Now, we define the probability measure \mathbb{P}_i for each $1 \leq i \leq n$, which implicitly defines the product probability measure \mathbb{P} using the definition of a product of probability spaces.

We now define $(\Omega_i, \Sigma_i, \mathbb{P}_i)$ as a product of probability spaces $(\Omega_i^1, \Sigma_i^1, \mathbb{P}_i^1) \times (\Omega_i^2, \Sigma_i^2, \mathbb{P}_i^2)$ which will separate sign from the choice of nonzero entries. Here, Ω_i^1 is the set $\{-1, 1\}$ and Ω_i^2 is the set of binary m -dimensional column vectors with exactly s nonzero entries. Notice there is a bijection $\Omega_i^1 \times \Omega_i^2 \rightarrow \Omega_i$ defined by $(\omega_i^1, \omega_i^2) \mapsto (\frac{1}{\sqrt{s}}\omega_i^1)\omega_i^2$. Now, we define $\mathbb{P}_i^1(-1) = \mathbb{P}_i^1(1) = 0.5$. We take \mathbb{P}_i^2 to be any probability measure that satisfies the following properties: We use the notation that for $\omega_i^2 \in \Omega_i^2$, $\omega_i^2(r)$ is the r th coordinate of ω_i^2 . First, for all $1 \leq r \leq m$, it must be true that $\sum_{\omega_i^2 \in \Omega_i^2 \text{ such that } \omega_i^2(r)=1} \mathbb{P}_i^2[\omega_i^2] = \frac{s}{m}$. Second, we assume that for any subset $S \subseteq [m]$:

$$\sum_{\omega_i^2 \in \Omega_i^2} \mathbb{P}_i^2[\omega_i^2] \left(\prod_{r \in S} \omega_i^2(r) \right) \leq \prod_{r \in S} \left(\sum_{\omega_i^2 \in \Omega_i^2} \mathbb{P}_i^2[\omega_i^2] \omega_i^2(r) \right).$$

This defines \mathbb{P}_i using the definition of a product of probability spaces and the bijection from $\Omega_i^1 \times \Omega_i^2 \rightarrow \Omega_i$ described above.

Let $\sigma_i : \Omega \rightarrow \{-1, 1\}$ be the random variable for the sign of the i th column. Let $\eta_{r,i} : \Omega \rightarrow \{0, 1\}$ be the indicator random variable for whether the (r, i) th entry is nonzero. Observe that the independence and negative correlation properties are recovered exactly.

B.2 Explanation of Proposition 10

As before, for $i \neq j$, we define $Q_{i,j} = \sum_{r=1}^m \eta_{r,i} \eta_{r,j}$. For some $r_1 \neq r_2 \neq \dots \neq r_s$, let $A_{r_1, \dots, r_s} \subseteq \Omega$ be the event that the nonzero entries in the i th column occur at r_1, \dots, r_s . Now, $(Q_{i,j} | A_{r_1, \dots, r_s})$ is a random variable defined by the restriction of the function $Q_{i,j} : \Omega \rightarrow \mathbb{R}$ to a function $A_{r_1, \dots, r_s} \rightarrow \mathbb{R}$ on the probability space $(A_{r_1, \dots, r_s}, 2^{A_{r_1, \dots, r_s}}, \mathbb{P}|_{A_{r_1, \dots, r_s}})$ where for $B \subseteq A_{r_1, \dots, r_s}$, we define $\mathbb{P}|_{A_{r_1, \dots, r_s}}(B)$ to be $\frac{\mathbb{P}(B)}{\mathbb{P}(A_{r_1, \dots, r_s})}$. This formally defines the random variable $(Q_{i,j} | \eta_{r_1,i} = \eta_{r_2,i} = \dots = \eta_{r_s,i} = 1)$ in Proposition 10, and observe that the proof proceeds by computing the distribution of this random variable. In this notation, the proposition shows that the random variables $(Q_{i,j} | A_{r_1, \dots, r_s})$ are independent on this new probability space.

B.3 Explanation of Lemma 13

For Lemma 13, the random variable $\left(\sum_{i < j \leq p} Q_{i,j} \right) | A_{r_1, \dots, r_s}$ is defined analogously on the restricted probability space. It follows from definition that this random variable is equivalent to the sum of random variables $\sum_{i < j \leq p} (Q_{i,j} | A_{r_1, \dots, r_s})$. Since we know that $(Q_{i,j} | A_{r_1, \dots, r_s})$ for $i < j \leq p$ are independent and have moments bounded by binomials by Proposition 10, this implies that we can apply Proposition 11 to $\sum_{i < j \leq p} (Q_{i,j} | A_{r_1, \dots, r_s})$ to obtain a bound on the p -norm $\mathbb{E} \left[\left(\sum_{i < j \leq p} Q_{i,j} | A_{r_1, r_2, \dots, r_s} \right)^p \right]^{1/p}$ which gives us a bound on the conditional p -norm $\mathbb{E} \left[\left(\sum_{i < j \leq p} Q_{i,j} \right)^p | A_{r_1, r_2, \dots, r_s} \right]^{1/p}$. Let's suppose this bound is α . Since the events $\{A_{r_1, \dots, r_s}\}_{r_1 \neq r_2 \neq \dots \neq r_s}$ form a disjoint partition of the probability space, we know by the law of total expectation that:

$$\alpha^p \geq \sum_{r_1 \neq r_2 \neq \dots \neq r_s} \mathbb{P}[A_{r_1, r_2, \dots, r_s}] \mathbb{E} \left[\left(\sum_{i < j \leq p} Q_{i,j} \right)^p | A_{r_1, r_2, \dots, r_s} \right] = \mathbb{E} \left[\left(\sum_{i < j \leq p} Q_{i,j} \right)^p \right],$$

which means that $\left\| \sum_{i < j \leq p} Q_{i,j} \right\|_p \leq \alpha$, so a uniform bound on the conditional p -norm implies a bound on the p -norm, which finishes the proof.

B.4 Explanation of Lemma 14

For Lemma 14, the random variable $\sum_{j > p, j \neq i} Q_{i,j} \sigma_j \mid A_{r_1, \dots, r_s}$ is defined analogously on the restricted probability space. It follows from definition that this random variable is equivalent to the sum of random variables $\sum_{j > p, j \neq i} (Q_{i,j} \sigma_j \mid A_{r_1, \dots, r_s}) = \sum_{j > p, j \neq i} (Q_{i,j} \mid A_{r_1, \dots, r_s})(\sigma_j \mid A_{r_1, \dots, r_s})$. By Proposition 10, we know that the random variables $(Q_{i,j} \mid A_{r_1, \dots, r_s})$ are independent for $j > p, j \neq i$ and have moments bounded by binomials. Moreover, from the construction of the probability space, we know that $(\sigma_j \mid A_{r_1, \dots, r_s})$ are independent for $j > p, j \neq i$ and are independent of all $(Q_{i,j} \mid A_{r_1, \dots, r_s})$, and we also know that $(\sigma_j \mid A_{r_1, \dots, r_s})$ is distributed as a Rademacher. This implies that we can apply Proposition 12 to obtain a bound on the p -norm $\mathbb{E} \left[\left| \sum_{j > p, j \neq i} (Q_{i,j} \mid A_{r_1, \dots, r_s})(\sigma_j \mid A_{r_1, \dots, r_s}) \right|^p \right]^{1/p}$ which we can view as a bound on the conditional p -norm $\mathbb{E} \left[\left| \sum_{i < j \leq p} Q_{i,j} x_j \sigma_j \right|^p \mid A_{r_1, r_2, \dots, r_s} \right]^{1/p}$. Let's suppose that this bound is α . Since the events $\{A_{r_1, \dots, r_s}\}_{r_1 \neq r_2 \neq \dots \neq r_s}$ form a disjoint partition of the probability space, we know by the law of total expectation that:

$$\alpha^p \geq \sum_{r_1 \neq r_2 \neq \dots \neq r_s} \mathbb{P}[A_{r_1, r_2, \dots, r_s}] \mathbb{E} \left[\left| \sum_{i < j \leq p} Q_{i,j} \sigma_j x_j \right|^p \mid A_{r_1, r_2, \dots, r_s} \right] = \mathbb{E} \left[\left| \sum_{i < j \leq p} Q_{i,j} \sigma_j x_j \right|^p \right],$$

which means that $\left\| \sum_{i < j \leq p} Q_{i,j} \sigma_j x_j \right\|_p \leq \alpha$, so a uniform bound on the conditional p -norm implies a bound on the p -norm, which finishes the proof.

C Latała's Moment Bounds and Proof Sketches

We sketch proofs of the upper bounds given in the following two lemmas, due to Latała. Full proofs of these lemmas can be found in [20].

► **Lemma 15** ([20]). *If $q \geq 2$ and X, X_1, \dots, X_n are independent symmetric random variables, then*

$$\left\| \sum_{i=1}^n X_i \right\|_q \simeq \inf \left\{ T > 0 \text{ such that } \sum_{i=1}^n \log \left(\mathbb{E} \left[\left(1 + \frac{X_i}{T} \right)^q \right] \right) \leq q \right\}.$$

► **Lemma 16** ([20]¹⁶). *If $1 \leq q \leq n$ and X, X_1, \dots, X_n are i.i.d nonnegative random variables, then*

$$\left\| \sum_{i=1}^n X_i \right\|_q \simeq \sup_{1 \leq t \leq q} \frac{q}{t} \left(\frac{n}{q} \right)^{1/t} \|X\|_t.$$

For a random variable X , we define

$$\phi_q(X) := \mathbb{E}[|1 + X|^q].$$

We begin with the following proposition that relates ϕ_q to the q -norm, which is useful in proving Lemma 16 and Lemma 15.

¹⁶This result was actually first due to S.J. Montgomery-Smith through a private communication with Latała. Nonetheless, it is also a corollary of a result in [20].

► **Proposition 17.** *If independent random variables X_1, \dots, X_n and value $q \geq 1$ satisfy the following inequality for any $T > 0$:*

$$\left(\left\| \sum_{i=1}^n \frac{X_i}{T} \right\|_q \right)^q \leq \prod_{i=1}^n \phi_q \left(\frac{X_i}{T} \right),$$

then

$$\left\| \left(\sum_{i=1}^n X_i \right) \right\|_q \lesssim \inf \left\{ T > 0 \text{ such that } \sum_{i=1}^n \log \left(\mathbb{E} \left[\left(1 + \frac{X_i}{T} \right)^q \right] \right) \leq q \right\}.$$

Proof. Suppose that $\sum_{i=1}^n \log \left(\mathbb{E} \left[\left(1 + \frac{X_i}{T} \right)^q \right] \right) \leq q$. Then,

$$\left\| \left(\sum_{i=1}^n X_i \right) \right\|_q = T \left\| \left(\sum_{i=1}^n \frac{X_i}{T} \right) \right\|_q \leq T \left(\prod_{i=1}^n \phi_q \left(\frac{X_i}{T} \right) \right)^{1/q} = T e^{\frac{1}{q} \log \left(\sum_{i=1}^n \phi_q \left(\frac{X_i}{T} \right) \right)} \lesssim T.$$

◀

The proofs of the upper bounds of Lemma 15 and Lemma 16 boil down to showing that the condition of Proposition 17 is satisfied. In Section C.1, we sketch a proof of the upper bound of Lemma 16. In Section C.2, we sketch a proof of the upper bound of Lemma 15.

C.1 Proof Sketch of Lemma 15 (Upper Bound)

We first state the following two propositions. The proof of these propositions are straightforward calculations and can be found in [20].

► **Proposition 18.** *If $q \geq 2$ and Y_1, Y_2, \dots, Y_n are independent symmetric random variables, then*

$$\phi_q \left(\sum_{i=1}^n Y_i \right) \leq \prod_{i=1}^n \phi_q(Y_i).$$

► **Proposition 19.** *If $q \geq 2$ and Y is a symmetric random variable, then*

$$\|Y\|_q^q \leq \phi_q(Y).$$

From Proposition 18 and Proposition 19, coupled with the fact that $X_1/T, X_2/T, \dots, X_n/T$, and $\sum_{i=1}^n X_i/T$ are independent symmetric random variables, it follows that the condition of Proposition 17 is satisfied.

C.2 Proof Sketch of Lemma 16 (Upper Bound)

We first sketch the proof of the upper bound of the following sublemma, which is analogous to Lemma 15:

► **Lemma 20** (Latała[20]). *If $q \geq 1$ and X, X_1, \dots, X_n are independent nonnegative random variables then*

$$\left\| \sum_{i=1}^n X_i \right\|_q \simeq \inf \left\{ T > 0 \text{ such that } \sum_{i=1}^n \log \left(\mathbb{E} \left[\left| 1 + \frac{X_i}{T} \right|^q \right] \right) \leq q \right\}.$$

The upper bound of Lemma 20 follows from the following propositions, coupled with the fact that $X_1/T, X_2/T, \dots, X_n/T$, and $\sum_{i=1}^n X_i/T$ are independent nonnegative random variables. The proofs of these propositions are straightforward calculations and can be found in [20].

► **Proposition 21.** *If $q \geq 1$ and Y_1, Y_2, \dots, Y_n are independent nonnegative random variables, then*

$$\phi_q \left(\sum_{i=1}^n X_i \right) \leq \prod_{i=1}^n \phi_q(X_i).$$

► **Proposition 22.** *If $q \geq 1$ and Y is a nonnegative random variable, then*

$$\|Y\|_q^q \leq \phi_q(Y).$$

Now, we describe how the upper bound of Lemma 20 implies the upper bound of Lemma 16. It suffices to show if we take

$$T = 2e \sup_{1 \leq t \leq q} \frac{q}{t} \left(\frac{n}{q} \right)^{1/t} \|X\|_t,$$

then

$$\sum_{i=1}^n \log \left(\mathbb{E} \left[\left| 1 + \frac{X_i}{T} \right|^q \right] \right) \leq q. \quad (4)$$

Since $q \leq n$, it follows that

$$\begin{aligned} \mathbb{E} \left[\left| 1 + \frac{X_i}{T} \right|^q \right] &= \frac{\mathbb{E}[X_i^q]}{T^q} + \sum_{k=0}^{q-1} \binom{q}{k} \frac{\mathbb{E}[X_i^k]}{T^k} \\ &\leq \frac{\mathbb{E}[X_i^q]}{\left(\left(\frac{n}{q} \right)^{1/q} \|X_i\|_q \right)^q} + \sum_{k=0}^{q-1} \left(\frac{qe}{k} \right)^k \frac{\mathbb{E}[X_i^k]}{\left(\frac{2eq\|X_i\|_k}{k} \right)^k} \\ &\leq \frac{q}{n} + 1 \\ &\leq e^{\frac{q}{n}} \end{aligned}$$

from which (4) follows.

D Proof of Proposition 12

The main ingredient in this proof is Lemma 15 (Latała's bound on moments of sums of symmetric random variables).

Proof of Proposition 12. Since the Y_i are independent random variables, we can apply Lemma 15 to obtain:

$$\left\| \sum_{k=1}^M Y_k y_k \sigma_k \right\|_q \lesssim \inf \left\{ T > 0 \mid \sum_{k=1}^M \log \left(\mathbb{E} \left[\left| 1 + \frac{Y_k \sigma_k y_k}{T} \right|^q \right] \right) \leq q \right\}.$$

Thus, it suffices to show

$$T \simeq \begin{cases} \frac{\sqrt{q}}{\log B} & \text{if } B \geq e \\ \frac{\sqrt{q}}{B} & \text{if } B < e \end{cases}$$

satisfies $\sum_{k=1}^M \log \left(\mathbb{E} \left[\left(1 + \frac{Y_k \sigma_k y_k}{T} \right)^q \right] \right) \leq q$. We see

$$\begin{aligned} \sum_{k=1}^M \log \left(\mathbb{E} \left[\left(1 + \frac{Y_k \sigma_k y_k}{T} \right)^q \right] \right) &= \sum_{k=1}^M \log \left(1 + \sum_{l=1}^q \binom{q}{l} \frac{(\mathbb{E}[Y_k \sigma_k^l]) y_k^l}{T^l} \right) \\ &= \sum_{k=1}^M \log \left(1 + \sum_{l=1}^{q/2} \binom{q}{2l} \frac{\|Y_k\|_{2l}^{2l} y_k^{2l}}{T^{2l}} \right) \\ &\leq \sum_{k=1}^M \log \left(1 + \sum_{l=1}^{q/2} \left(\frac{qe}{2l} \right)^{2l} \left(\frac{\|Y_k\|_{2l} y_k}{T} \right)^{2l} \right) \end{aligned}$$

By the bound on moments of binomial random variables in Proposition 11, we know if $\frac{l}{\alpha \max(N, l)} \geq B \geq e$ that there exists a universal constant C such that $\|Q_{i,j}\|_{2l} \leq \frac{2lC}{\log B}$. Thus, we obtain

$$\begin{aligned} \sum_{k=1}^M \log \left(\mathbb{E} \left[\left(1 + \frac{Y_k \sigma_k y_k}{T} \right)^q \right] \right) &\leq \sum_{k=1}^M \log \left(1 + \sum_{l=1}^{q/2} \left(\frac{qe}{2l} \right)^{2l} \left(\frac{2lC y_k}{T \log B} \right)^{2l} \right) \\ &\leq \sum_{k=1}^M \log \left(1 + \sum_{l=1}^{q/2} \left(\frac{qeC y_k}{T \log B} \right)^{2l} \right). \end{aligned}$$

Since $|y_k| \leq \frac{1}{\sqrt{q}}$, if we set $T = \frac{2eC\sqrt{q}}{\log B}$, then we obtain

$$\sum_{k=1}^M \log \left(1 + \sum_{l=1}^{q/2} \left(\frac{\sqrt{q} y_k}{2} \right)^{2l} \right) \leq \sum_{k=1}^M \log \left(1 + (\sqrt{q} y_k)^2 \sum_{l=1}^{q/2} \left(\frac{1}{2} \right)^{2l} \right).$$

This can be bounded by

$$\sum_{k=1}^M \log \left(1 + (\sqrt{q} y_k)^2 \right) = \sum_{k=1}^M \log \left(1 + q y_k^2 \right) \leq \sum_{i=1}^n q y_k^2 \leq q$$

as desired. An analogous argument shows that if $B < e$, we can set $T = \frac{2eC\sqrt{q}}{B}$. ◀

E Proof of Proposition 11

The main tool that we use in this proof is Lemma 16 (Latala's bound on moments of sums of i.i.d nonnegative random variables).

Proof of Proposition 11. Notice that it suffices to obtain an upper bound on $\|X\|_q$ for all $N \geq q$. (Since $\|X\|_q$ is an increasing function of N , an upper bound on $\|X\|_q$ at $N = q$ is also an upper bound on $\|X\|_q$ for all $N < q$). For the rest of the proof, we assume $N \geq q$.

Notice X has the same distribution as $\sum_{j=1}^N Z_j$ where Z, Z_1, \dots, Z_N are i.i.d Bernoulli random variables with expectation α . Since $\|Z\|_t = \alpha^{1/t}$, we know by Lemma 16,

$$\begin{aligned} \|X\|_q &\simeq \sup_{1 \leq t \leq q} \frac{q}{t} \left(\frac{N}{q} \right)^{1/t} \alpha^{1/t} \\ &= \sup_{1 \leq t \leq q} \frac{q}{t} \left(\frac{1}{B} \right)^{1/t} \end{aligned}$$

At $t = 1$, this quantity is equal to $\frac{q}{B}$, and at $t = q$, this quantity is equal to $(\frac{1}{B})^{1/q} = e^{\log(1/B)/q}$. The only $t \in \mathbb{R}$ for which this quantity has derivative 0 is $t = \log B$. Notice that $1 \leq \log B \leq q$ if and only if $e \leq B \leq e^q$. Thus

$$\|X\|_q \simeq \begin{cases} \max(\frac{q}{B}, \frac{q}{\log B}, e^{\log(1/B)/q}) & \text{if } e \leq B \leq e^q \\ \max(\frac{q}{B}, e^{\log(1/B)/q}) & \text{if } B < e \text{ or if } B > e^q. \end{cases}$$

For $B \geq e$, we want to show $\|X\|_q \lesssim q/\log B$. Since $\log B > 0$, we see $e^{\log(1/B)/q} = e^{-\log B/q} \leq q/\log B$ and $q/B \leq q/\log B$.

For $B < e$, we want to show $\|X\|_q \lesssim q/B$. Since $\frac{1}{B} > \frac{1}{e}$, we see $e^{\log(1/B)/q} = (\frac{1}{B})^{1/q} \leq \frac{e}{B} \lesssim \frac{q}{B}$. \blacktriangleleft

F Weakness of bound on $\|Z\|_p$ from Equation (5)

Like in Section 4, we view the random variable Z as a quadratic form $\frac{1}{s}\sigma^T A\sigma$, where σ an n -dimensional vector of independent Rademachers and A is a symmetric, zero-diagonal $n \times n$ matrix where the (i, j) th entry (for $i \neq j$) is $x_i x_j \sum_{r=1}^m \eta_{r,i} \eta_{r,j} = Q_{i,j} x_i x_j$. Applying the Hanson-Wright bound followed by an expectation over the η values yields

$$\|\sigma^T A\sigma\|_p \lesssim \left\| \sqrt{p} \sqrt{\sum_{i=1}^n \sum_{j \leq n, j \neq i} Q_{i,j}^2 x_i^2 x_j^2} + p \sup_{\|y\|_2=1} \left| \sum_{i=1}^n \sum_{j \leq n, j \neq i} Q_{i,j} x_i x_j y_i y_j \right| \right\|_p =: U_p. \quad (5)$$

We show that the vector $x = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0] \in \mathbb{R}^n$ forces U_p to be too large to yield the optimal m value, thus proving that the Hanson-Wright bound does not provide a sufficiently tight bound on $\|Z\|_p$ to achieve Theorem 3. The main ingredient in our proof is the following lemma, which we prove in subsection C.1:

► **Lemma 23.** *For every column $1 \leq i \leq n$, suppose that the random variables $\{\eta_{r,i}\}_{r \in [m], i \in [n]}$ have the distribution defined by uniformly choosing exactly s of the variables per column. If $x = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0]$, $p < s$ and $B = m/s^2 \leq \frac{e^p}{p}$, then*

$$U_p \simeq \begin{cases} \frac{p^2}{\log B p} & \text{if } B \geq \frac{e}{p} \\ \frac{p}{B} & \text{if } B < \frac{e}{p}. \end{cases}$$

We can obtain bounds on s and m from Lemma 23 via Markov's inequality. We disregard the case where $B \geq \frac{e^p}{p}$, since this case would yield a value for m that is not polynomial in $\log(1/\delta)$. If $B < e/p$, then it follows that $s = \Theta(\varepsilon^{-1} B^{-1} \log(1/\delta)) = \Omega(\varepsilon^{-1} \log^2(1/\delta))$ and $m = \Theta(\varepsilon^{-2} B^{-1} \log^2(1/\delta)) = \Omega(\varepsilon^{-2} \log^3(1/\delta))$. If $B \geq e/p$, then it follows that $s = \Theta(\varepsilon^{-1} p^2 / \log(Bp)) = \Omega(\varepsilon^{-1} \log(1/\delta))$ and $m = \Theta(\varepsilon^{-2} p^4 B / \log^2(Bp)) = \Omega(\varepsilon^{-2} \log^3(1/\delta))$. These bounds on m incur an extra $\log(1/\delta)$ factor, and thus the Hanson-Wright bound is too weak for this setting. Now, it suffices to prove Lemma 23, which we do in the next section.

F.1 Proof of Lemma 23

In this section, we assume that $x = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0]$ and that the random variables $\{\eta_{r,i}\}_{r \in [m], i \in [n]}$ have the distribution defined by uniformly choosing exactly s of the variables per column. We first show the following computation of $\|Q_{i,j}\|_p$.

► **Proposition 24.** *Assume that the random variables $\{\eta_{r,i}\}_{r \in [m], i \in [n]}$ have the distribution defined by uniformly choosing exactly s of the variables per column. Then, if $p < s$ and $X \sim \text{Bin}(s, s/m)$, we have that $\|Q_{i,j}\|_p \simeq \|X\|_p$.*

Proof. We condition on the event that the nonzero locations in column i are at r_1, r_2, \dots, r_s . Notice that the random variable $(Q_{i,j} \mid \eta_{r_1,i} = \eta_{r_2,i} = \dots = \eta_{r_s,i} = 1)$ is distributed as $Z_{r_1} + Z_{r_2} + \dots + Z_{r_s}$ where Z_{r_k} is an indicator for the k th entry in the j th column being nonzero. Let Z'_{r_k} for $1 \leq k \leq s$ be i.i.d random variables distributed as $\text{Bern}(s/m)$. Now, observe that

$$\mathbb{E}[(Z_{r_1} + Z_{r_2} + \dots + Z_{r_s})^p] = \sum_{\substack{0 \leq t_1, t_2, \dots, t_s \leq p \\ t_1 + t_2 + \dots + t_s = p}} \mathbb{E}[\prod_{i=1}^s Z_{r_i}^{t_i}] = \sum_{\substack{0 \leq t_1, t_2, \dots, t_s \leq p \\ t_1 + t_2 + \dots + t_s = p}} \mathbb{E}[\prod_{i|t_i > 0} Z_{r_i}].$$

Notice that $\mathbb{E}[(Z'_{r_1} + Z'_{r_2} + \dots + Z'_{r_s})^p] = \sum_{0 \leq t_1, t_2, \dots, t_s \leq p, t_1 + t_2 + \dots + t_s = p} \mathbb{E}[\prod_{i|t_i > 0} Z'_{r_i}]$. Thus, it suffices to compare $\mathbb{E}[\prod_{i|t_i > 0} Z_{r_i}]$ and $\mathbb{E}[\prod_{i|t_i > 0} Z'_{r_i}]$. We see that $\mathbb{E}[\prod_{i|t_i > 0} Z_{r_i}] = \left(\frac{s}{m}\right)^{|\{i|t_i > 0\}|}$. Since $p < s$, we see that $\mathbb{E}[\prod_{i|t_i > 0} Z_{r_i}] = \prod_{j=0}^{|\{i|t_i > 0\}|-1} \frac{s-j}{m-j}$. It is not difficult to verify that this ratio is bounded by $2^{O(p)}$ as desired, so

$$\frac{\mathbb{E}[(Q_{i,j} \mid \eta_{r_1,i} = \eta_{r_2,i} = \dots = \eta_{r_s,i} = 1)^p]}{\mathbb{E}[X^p]} = \frac{\mathbb{E}[(Z_{r_1} + Z_{r_2} + \dots + Z_{r_s})^p]}{\mathbb{E}[X^p]} \geq 2^{-O(p)}.$$

Now, by the law of total expectation, we know that

$$\frac{\mathbb{E}[Q_{i,j}^p]}{\mathbb{E}[X^p]} \geq 2^{-O(p)}$$

as desired. ◀

We now prove the following relation between U_p and $\|Q_{1,2}\|_p$:

► **Lemma 25.** *Assume the notation and restrictions above. Then $U_p \simeq p \|Q_{1,2}\|_p$.*

Proof of Lemma 25. For ease of notation, we define

$$S_1 := p \sup_{\|y\|_2=1} \left| \sum_{i=1}^n \sum_{j \leq n, j \neq i} Q_{i,j} x_i x_j y_i y_j \right|$$

$$S_2 := \sqrt{p} \sqrt{\sum_{i=1}^n \sum_{j=1}^n Q_{i,j}^2 x_i^2 x_j^2}.$$

Our goal is to calculate $U_p = \|S_1 + S_2\|_p$. We make use of the following upper and lower bounds on $\|S_1 + S_2\|_p$:

$$\left| \|S_1\|_p - \|S_2\|_p \right| \leq \|S_1 - S_2\|_p \leq \|S_1 + S_2\|_p \leq \|S_1\|_p + \|S_2\|_p. \quad (6)$$

In order to compute $\|S_1\|_p$ and $\|S_2\|_p$, we first compute $\|S_1\|_p$ and $\|S_2\|_p$. For our choice of x , notice

$$\|S_1\|_p \simeq p \left\| \sup_{\|y\|_2=1} |Q_{1,2} y_1 y_2| \right\|_p \simeq p \|Q_{1,2}\|_p$$

$$\|S_2\|_p \simeq \sqrt{p} \left\| \sqrt{Q_{1,2}^2} \right\|_p = \sqrt{p} \|Q_{1,2}\|_p.$$

From these bounds, coupled with (6), it follows that $\|U\|_p \simeq p \|Q_{1,2}\|_p$ as desired. ◀

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We now show Lemma 23 follows from Lemma 25 and Proposition 24.

Proof of Lemma 23. After applying Lemma 25, it suffices to calculate $\|Q_{1,2}\|_p$. It follows from Proposition 24 that $\|Q_{1,2}\|_p \simeq \|X\|_p$ where X is distributed as $\text{Bin}(s, s/m)$. Now, the following calculation $\|X\|_p$ for $p < s$ and $B = m/s^2 \leq \frac{e^p}{p}$ follows from the lower and upper bounds of Lemma 16 (Latała's bound on moments of sums of i.i.d nonnegative random variables):

$$\|X\|_p \simeq \begin{cases} \frac{p}{\log Bp} & \text{if } B \geq \frac{e}{p} \\ \frac{1}{B} & \text{if } B < \frac{e}{p} \end{cases}.$$

From this, Lemma 23 follows. ◀