# Simple Analysis of Sparse, Sign-Consistent JL

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RANDOM 2019

# Linear dimensionality reduction: $\ell_2$-to-$\ell_2$

**Informal goal**: Project vectors in $\mathbb{R}^n$ to $\mathbb{R}^m$ (for $m \ll n$) with a linear map while "preserving geometry" (i.e. $\|f(x) - f(y)\|_2 \approx \|x - y\|_2$).

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Many applications:

- Feature hashing (Weinberger et al. '09, Dahlgaard et al. '17, Freksen et al. '18, etc.)
- Numerical linear algebra (Clarkson and Woodruff '12, Nelson and Nguyen '14, etc.)
- Approximate nearest neighbors (Ailon and Chazelle '09, etc.)
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- Compression in the brain (Allen-Zhu, Gelashvili, Micali, Shavit '15)

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Use a probability distribution $\mathcal{M}$ over linear maps $\mathbb{R}^n \to \mathbb{R}^m$ ($m \ll n$).

**Geometry-preserving property:** for *each* $x \in \mathbb{R}^n$

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Fundamental result of linear dimensionality reduction:

## Lemma (Distributional Johnson-Lindenstrauss Lemma)

*Can obtain $m = \Theta(\epsilon^{-2}\log(1/\delta))$ using $\mathcal{M}$ with i.i.d gaussian entries.*

This dimension is actually optimal for any distribution over linear maps (Kane et al. '11, Jayram and Woodruff '11).

Optimality for $N$-point version (Larsen and Nelson '17)

# Application to information compression in the brain

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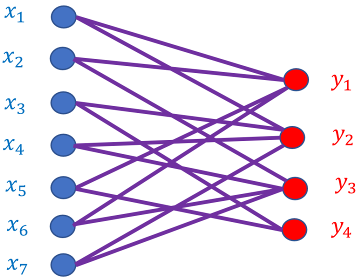
Convergent pathways compress information w/o losing the ability to perform computations.

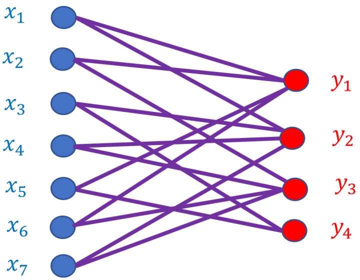



A model (Ganguli, Sompolinsky '12)

- Source information: $x \in \mathbb{R}^n$
- Target information: $y \in \mathbb{R}^m$
- Synaptic connections: a random matrix $M \in \mathbb{R}^{m \times n}$

# Biological Constraints on Supp($\mathcal{M}$)

(As per Ganguli and Sompolinsky's model)



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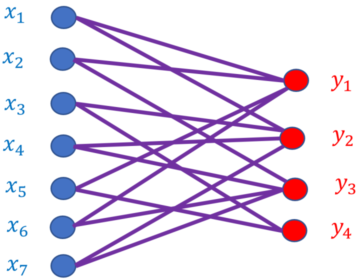


**Sparsity:** every column has $\leq s$ nonzero entries.

- Neurons connected to few post-synaptic neurons

**Sign-consistency:** in each column, nonzero entries are *all positive* or *all negative*

- Neurons are excitatory or inhibitory

# JL for sparse matrices

Sparsity is also more generally useful for reducing projection time.

## Informal Construction (Sparse JL)

*Uniformly choose $s$ nonzero entries per column; i.i.d signs for nonzero entries*

Can set $m = \Theta(\epsilon^{-2} \log(1/\delta))$ and $s = \Theta(\epsilon^{-1} \log(1/\delta)$ (Kane and Nelson, J. ACM '12)

Can set $m = \min(2\epsilon^{-2}/\delta, \Theta(\epsilon^{-2} \log(1/\delta) B))$ and $s = \Theta(\epsilon^{-1} \log(1/\delta)/ \log B)$ (Cohen, SODA '16)

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*$\mathcal{M}$ is defined so the $(r, i)$th entry is $\sigma_i \eta_{r,i} / \sqrt{s}$ where:*

- *$\sigma_i$ are i.i.d. Rademachers (random signs)*
- *$\eta_{r,i}$ are $\{0, 1\}$ rvs s.t. $\sum_{r=1}^{m} \eta_{r,i} = s$ and w/ mild assumptions*

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Can set $m = \Theta(\epsilon^{-2} \log^2(1/\delta))$, $s = \Theta(\epsilon^{-1} \log(1/\delta))$ (Allen-Zhu, Gelashvili, Micali, and Shavit, PNAS '15)

# This work

Simplify and generalize the analysis of sparse, sign-consistent JL.

## Theorem (Informal)

*For any* $e \leq B \leq 1/\delta$*, can set* $m = \Theta(\epsilon^{-2} \log^2(1/\delta) B / \log^2(B))$ *and* $s = \Theta(\epsilon^{-1} \log(1/\delta) / \log B)$ *for sparse, sign-consistent JL.*

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Remainder of the talk will focus on the proof method.

# High-level approach

Need to show $\mathbb{P}_{M \in \mathcal{M}}[(1-\epsilon)\left\|x\right\|_2 \leq \left\|Mx\right\|_2 \leq (1+\epsilon)\left\|x\right\|_2] > 1-\delta$.

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For sparse, sign-consistent JL:

$$Z := \left\|Mx\right\|_2^2 - 1 = \frac{1}{s} \sum_{i \neq j} \sum_{r=1}^{m} \sigma_i \sigma_j \eta_{r,i} \eta_{r,j} x_i x_j.$$

# Analyzing the moments of $Z$

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My key ingredient: more precise quadratic form bounds

# Expressing $Z$ as a Rademacher quadratic form

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$\implies \mathbb{E}[Z^p] = \mathbb{E}_\eta \left[ \mathbb{E}_\sigma \left[ \left( \sigma^T A_\eta \sigma \right)^p \right] \right].$

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Thus, we need a Rademacher-specific bound for $\mathbb{E}_\sigma [(\sigma^T A_\eta \sigma)^p]$.

# A digression on Rademachers vs. gaussians for linear forms

$\|Y\|_p := (\mathbb{E}[Y^p])^{1/p}$; $\sigma_i$ i.i.d. Rademachers; $g_i$ i.i.d gaussians;
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**Khintchine bound**: $\|\sum_{i=1}^n a_i \sigma_i\|_p \lesssim \|\sum_{i=1}^n a_i g_i\|_p \sim \sqrt{p}\sqrt{\sum_{i=1}^n a_i^2}$

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Turns out $\left\|\sum_{i=1}^n a_i \sigma_i\right\|_p \sim \left|\sum_{i=1}^p a_i\right| + \sqrt{p}\sqrt{\sum_{i>p} a_i^2}$ (Hitzchenko '93).

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## Lemma

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(Tight bound on $\left\| \sigma^T A \sigma \right\|_p$ (Latała '99) messy when $A$ a *random matrix*.)

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**"Understanding Sparse JL for Feature Hashing" (NeurIPS 2019)**

I study sparse JL on feature vectors.

- Model: limit to vectors $x$ with "small" $\ell_\infty$-to-$\ell_2$ norm ratio
- $s = 1$ understood (Weinberger et al '09, Dahlgaard et al. '17, Freksen et al. '18, etc.)

My main result: Generalization to $s > 1$.

- Tight tradeoff between $\ell_\infty$-to-$\ell_2$ ratio, $s$, $m$, $\epsilon$, and $\delta$ for sparse JL
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- $\implies$ Even (small) $s > 1$ can be much better than $s = 1$.

Similarly unclear how to adapt combinatorics; gaussian bounds too weak.

Tractable Rademacher-specific bounds are the key technical tool.

# Conclusion

- Simplified and generalized the analysis of sparse, sign-consistent JL (Allen-Zhu, Gelashvili, Micali, Shavit '15).
- Specifically obtained dimensionality-sparsity tradeoffs $m = \Theta(\epsilon^{-2}\log^2(1/\delta)B/\log^2(B))$ and $s = \Theta(\epsilon^{-1}\log(1/\delta)/\log B)$.
- Introduced a simple moment bound for Rademacher quadratic forms which enables a simpler analysis of sparse, sign-consistent JL, and could be of broader use.